

SPECTRAL PROPERTIES OF SCHRÖDINGER OPERATORS ARISING IN THE STUDY OF QUASICRYSTALS

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ABSTRACT. We survey results that have been obtained for self-adjoint operators, and especially Schrödinger operators, associated with mathematical models of quasicrystals. After presenting general results that hold in arbitrary dimensions, we focus our attention on the one-dimensional case, and in particular on several key examples. The most prominent of these is the Fibonacci Hamiltonian, for which much is known by now and to which an entire section is devoted here. Other examples that are discussed in detail are given by the more general class of Schrödinger operators with Sturmian potentials. We put some emphasis on the methods that have been introduced quite recently in the study of these operators, many of them coming from hyperbolic dynamics. We conclude with a multitude of numerical calculations that illustrate the validity of the known rigorous results and suggest conjectures for further exploration.

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1. INTRODUCTION

The area of mathematical quasicrystals is fascinating due to its richness and very broad scope. One perspective that leads to a rich theory is the question of how well quantum wave packets can travel in a quasicrystalline medium. This, in turn, leads to a study of the time-dependent Schrödinger equation governed by a potential that reflects the aperiodic order of the environment to which the quantum state is exposed.

This survey paper describes the current “state of the art” of mathematical results concerning quantum transport in mathematical quasicrystal models. We will describe the (Schrödinger) operators that are typically considered in this context, as well as the spectral and quantum dynamical properties of these operators that are relevant to our basic motivating question. Few results hold without further assumptions. After presenting these general results, we will therefore specialize our class of operators in various ways. On the one hand, additional general results follow from passing to a one-dimensional setting, most notably some specific consequences of Kotani theory. On the other hand, there is much interest in certain central examples, such as the Fibonacci model, and more generally the Sturmian case or potentials generated by primitive substitutions. For these special cases, many more results are known, most of which will be described in detail here. Another purpose of this survey is to highlight new tools that have recently been introduced in the study of these models that have led to very fine quantitative results in the Fibonacci case and beyond.

The structure of the paper is as follows. In Section 2 we describe the operators that are typically considered in the case of general dimension. In Section 3 we present the results known to hold in this very general setting. Next, in Section 4, we pass to the one-dimensional situation, where the operators are slightly redefined to conform with the bulk of the literature. Additional general results are described in this scenario. These range from Kotani theory, via proofs of zero-measure spectrum and hence absence of absolutely continuous spectrum based on Kotani theory, to proofs of absence of point spectrum and hence purely singular continuous spectrum based on Gordon’s lemma. We also discuss how these general results can be applied to several classes of examples. Section 5 is devoted to the special case of the Fibonacci Hamiltonian. This is the most prominent one-dimensional quasicrystal model; in addition to the general results mentioned before, there are fine estimates

of the local and global dimensions of the spectrum and the density of states measure, as well as the optimal Hölder exponent of the integrated density of states, at least in the regimes of small or large coupling. For this model the question about quantum transport behavior has very satisfactory answers, at least in the regime of large coupling. All these results, along with comments on their proofs, are presented in this section. Then, in Section 6, we discuss how the approach and the results may be extended from the Fibonacci case to the more general Sturmian case. We present and discuss numerical results in Section 7. Finally, Section 8 lists and discusses a number of open problems that are suggested by the existing results.

Related matters have been surveyed earlier in [10, 26, 41, 44, 147].

2. SCHRÖDINGER OPERATORS ARISING IN THE STUDY OF QUASICRYSTALS

In this section we describe self-adjoint operators that have been studied in the context of quasicrystals. There is a clear distinction between the case of one space dimension and the case of higher space dimensions. While the geometry of the quasicrystal model plays an important role in higher dimensions, this is not the case for one-dimensional models. In the former case, this leads to a dependence of the underlying Hilbert space on the realization of the model. In particular, as one typically embeds any such realization in a family of realizations, this leads to some technical issues that need to be addressed mathematically. In the latter case, on the other hand, one usually works in a universal Hilbert space and the aperiodic order features are solely reflected by the potential of the operator. This allows one to invoke the standard theory of ergodic Schrödinger operators with a fixed Hilbert space. In this survey we will present the known results in the settings in which they have been obtained, which we now describe.

Quasicrystals are commonly modeled either by Delone sets in Euclidean space or by tilings of Euclidean space. In fact, any such Delone set or tiling is embedded in a Delone or tiling dynamical system, which is obtained by considering the set of translates and then taking the closure of this set with respect to a suitable topology. This orbit closure is called the hull of the initial Delone set or tiling. The dynamics is then given by the natural action of the Euclidean space on the hull through translations. For this dynamical system, one then identifies ergodic measures, and they are typically unique. While each element of the hull gives rise to a Schrödinger operator, it is the ergodic framework that allows one to prove statements that hold for almost all such operators, as opposed to results for a single operator. Occasionally, it is then even possible to extend results that hold for almost all elements of the hull to all elements of the hull by approximation.

Delone sets and tilings are in some sense dual and hence equivalent to each other.¹ For definiteness, we will consider a framework based on Delone sets. Let us fix a dimension $d \geq 1$. The Euclidean norm on \mathbb{R}^d will be denoted by $|\cdot|$. We denote by $B(x, r)$ the closed ball in \mathbb{R}^d that is centered at x and has radius r .

Definition 2.1. *A set $\Lambda \subset \mathbb{R}^d$ is called a Delone set if there are $r, R > 0$ such that*

$$B(x, r) \cap \Lambda = \{x\} \quad \forall x \in \Lambda$$

¹One can go back and forth between these two settings by decorations and the Voronoi construction.

and

$$B(x, R) \cap \Lambda \neq \emptyset \quad \forall x \in \mathbb{R}^d.$$

Thus for a Delone set we have a lower bound on the separation between points of the set, and an upper bound for the size of “holes” in the set. One also says that a Delone set is uniformly discrete and relatively dense. Since Delone sets are closed and we want to take orbit closures, let us now define the underlying topology on $\mathcal{F}(\mathbb{R}^d)$, the closed subsets of \mathbb{R}^d , following [111].

Definition 2.2. For $k \in \mathbb{Z}_+$ and $F, G \in \mathcal{F}(\mathbb{R}^d)$, we define

$$d_k(F, G) := \inf \{ \varepsilon > 0 : F \cap B(0, k) \subset U_\varepsilon(G) \text{ and } G \cap B(0, k) \subset U_\varepsilon(F) \} \cup \{1\}.$$

For $k \in \mathbb{Z}_+$, $\varepsilon > 0$, and $F \in \mathcal{F}(\mathbb{R}^d)$, we define

$$U_{\varepsilon, k}(F) := \{G \in \mathcal{F}(\mathbb{R}^d) : d_k(F, G) < \varepsilon\}.$$

The topology on $\mathcal{F}(\mathbb{R}^d)$ with neighborhood basis $\{U_{\varepsilon, k}(F)\}$ will be called the natural topology and denoted by τ_{nat} .

Proposition 2.3. (a) Translations are continuous with respect to τ_{nat} .
 (b) $\mathcal{F}(\mathbb{R}^d)$ endowed with τ_{nat} is compact.
 (c) τ_{nat} is metrizable.

See [111] for a metric that induces τ_{nat} .

Definition 2.4. Let Ω be a set of Delone sets and denote by T the translation action of \mathbb{R}^d , that is, $T_t x = x + t$. The pair (Ω, T) is a Delone dynamical system if Ω is invariant under T and closed in the natural topology.

Definition 2.5. A Delone dynamical system is said to be of finite local complexity if for every radius $s > 0$, there is a uniform upper bound on the number of different patterns one can observe in ω intersected with a ball of radius s . Here, ω ranges over Ω , the center of the ball ranges over \mathbb{R}^d , and patterns are understood modulo translations.

Definition 2.6. Suppose (Ω, T) is a Delone dynamical system of finite local complexity. A family $\{A_\omega\}_{\omega \in \Omega}$ of bounded operators $A_\omega : \ell^2(\omega) \rightarrow \ell^2(\omega)$ is said to have finite range if there is $s > 0$ such that for all $\omega \in \Omega$, $A_\omega(x, y)$ only depends on the pattern of ω in the s -neighborhood of x and y , and $A_\omega(x, y) = 0$ whenever $|x - y| \geq s$.

The class of operators so defined encompasses all discrete operators that are usually considered as quantum Hamiltonians in the context of multi-dimensional quasicrystals. In one dimension, however, it is customary to realign points as a lattice (i.e., \mathbb{Z}) and to encode the geometry in the matrix elements of the operator. Even more specifically, one focuses on nearest neighbor interactions and hence obtains a tridiagonal matrix in the standard basis of $\ell^2(\mathbb{Z})$. Much of the mathematical literature focuses on the discrete Schrödinger case, where the terms on the first off-diagonals are all equal to one and the quasicrystalline structure of the environment is reflected in the terms on the diagonal. That is, the family of discrete Schrödinger operators one then considers is of the form $\{H_\omega\}_{\omega \in \Omega}$, where Ω is typically a subshift over a finite alphabet, and for $\omega \in \Omega$, H_ω acts on vectors from $\ell^2(\mathbb{Z})$ as follows,

$$[H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n).$$

Here the potential V_ω is given by $V_\omega(n) = f(T^n\omega)$, where $T : \Omega \rightarrow \Omega$ is the standard shift transformation and f is at least continuous, usually locally constant, and often just depends on a single entry of the sequence ω .

3. GENERAL RESULTS IN ARBITRARY DIMENSION

3.1. Spectrum and Spectral Types. One of the fundamental results for the families of operators introduced above is the almost sure constancy of the spectrum and the spectral type with respect to an ergodic measure μ associated with (Ω, T) . This follows from the covariance condition

$$A_{\omega+t} = U_t A_\omega U_t^*, \quad \omega \in \Omega, \quad t \in \mathbb{R}^d,$$

where $U_t : \ell^2(\omega) \rightarrow \ell^2(\omega+t)$ is the unitary operator induced by translation by t , along with the definition of ergodicity applied to traces of spectral projections in the usual way. This establishes the following result; compare [111].

Theorem 3.1. *Suppose (Ω, T) is a Delone dynamical system of finite local complexity and μ is an ergodic measure. Let $\{A_\omega\}_{\omega \in \Omega}$ be a family of bounded self-adjoint operators of finite range. Then there exist $\Sigma, \Sigma_{\text{pp}}, \Sigma_{\text{sc}}, \Sigma_{\text{ac}}$ and a subset $\Omega_0 \subseteq \Omega$ of full μ -measure such that for every $\omega \in \Omega_0$, we have $\sigma(A_\omega) = \Sigma$, $\sigma_{\text{pp}}(A_\omega) = \Sigma_{\text{pp}}$, $\sigma_{\text{sc}}(A_\omega) = \Sigma_{\text{sc}}$, and $\sigma_{\text{ac}}(A_\omega) = \Sigma_{\text{ac}}$.*

3.2. Existence of the Integrated Density of States. Suppose $\{A_\omega\}_{\omega \in \Omega}$ is a family of bounded self-adjoint operators of finite range. For $Q \subset \mathbb{R}^d$ bounded, the restriction $A_\omega|_Q$ defined on $\ell^2(Q \cap \omega)$ has finite rank. Therefore,

$$n(A_\omega, Q)(E) := \#\{\text{eigenvalues of } A_\omega|_Q \text{ that are } \leq E\}$$

is finite and $E \mapsto \frac{1}{|Q|}n(A_\omega, Q)(E)$ is the distribution function of the measure $\rho_Q^{A_\omega}$ defined by

$$\int \varphi d\rho_Q^{A_\omega} = \frac{1}{|Q|} \text{Tr}(\varphi(A_\omega|_Q)), \quad \varphi \in C_b(\mathbb{R}).$$

For $s > 0$ and $Q \subseteq \mathbb{R}^d$, denote by $\partial_s Q$ the set of points in \mathbb{R}^d whose distance from the boundary of Q is bounded by s . A sequence $\{Q_k\}$ of bounded subsets of \mathbb{R}^d is called a van Hove sequence if $\frac{\text{vol}(\partial_s Q_k)}{\text{vol}(Q_k)} \rightarrow 0$ as $k \rightarrow \infty$ for every $s > 0$.

The following result was shown in [112] (compare also the earlier papers [87, 88]).

Theorem 3.2. *Suppose (Ω, T) is a strictly ergodic Delone dynamical system of finite local complexity. Let $\{A_\omega\}_{\omega \in \Omega}$ be a family of bounded self-adjoint operators of finite range and let $\{Q_k\}$ be a van Hove sequence. Then, as $k \rightarrow \infty$, the distributions of $E \mapsto \rho_{Q_k}^{A_\omega}((-\infty, E])$ converge to the distribution $E \mapsto \rho^A((-\infty, E])$ with respect to $\|\cdot\|_\infty$, and the convergence is uniform in $\omega \in \Omega$.*

In fact, the limiting distribution can be given in closed form; see [112]. It is called the integrated density of states, and the associated measure is called the density of states measure. The remarkable feature of this result is the strength of the convergence, in that the distribution functions converge uniformly in the $\|\cdot\|_\infty$ topology. This is of particular interest in cases when the limiting distribution function has jumps. The next subsection shows that the latter phenomenon may actually happen.

3.3. Locally Supported Eigenfunctions and Discontinuities of the IDS. In dimensions strictly greater than one, the local structure of a Delone set may be chosen such that suitable finite range operators have finitely supported eigenfunctions at a suitable energy. If these local configurations occur sufficiently regularly, it follows that the energy in question will be a point of discontinuity of the integrated density of states. This observation may now be supplemented in two ways. On the one hand, a given Delone dynamical system may be transformed into one that is equivalent to the original one in the sense of mutual local derivability, which does have the required local configurations. On the other hand, any discontinuity of the integrated density of states must arise in this way, that is, through the regular occurrence of finitely supported eigenfunctions at the energy in question. These issues were discussed in the paper [102]. Let us state the results from that paper precisely.

Theorem 3.3. *Suppose (Ω, T) is a strictly ergodic Delone dynamical system of finite local complexity. Let $\{A_\omega\}_{\omega \in \Omega}$ be a family of bounded self-adjoint operators of finite range. Then there exist (Ω', T) and $\{A'_\omega\}_{\omega \in \Omega'}$ such that (Ω, T) and (Ω', T) are mutually locally derivable and A'_ω has locally supported eigenfunctions with the same eigenvalue for every $\omega \in \Omega'$. Moreover, A'_ω can be chosen to be the nearest neighbor Laplacian of a suitable graph.*

Theorem 3.4. *Suppose (Ω, T) is a strictly ergodic Delone dynamical system of finite local complexity. Let $\{A_\omega\}_{\omega \in \Omega}$ be a family of bounded self-adjoint operators of finite range. Then $E \in \mathbb{R}$ is a point of discontinuity of ρ^A if and only if there exists a locally supported eigenfunction of A_ω to E for one (equivalently, all) $\omega \in \Omega$.*

4. GENERAL RESULTS IN ONE DIMENSION

Starting with this section, we will focus on the case of one space dimension. Far more rigorous results are known for this special case than for the general case. In particular, much is known about the structure of the spectrum as a set, as well as the type of the spectral measures.

As mentioned above, in the one dimensional setting one typically passes to a somewhat different choice of the model. Thus, for definiteness, we will restrict our attention in much of the remainder of this paper to the following scenario. We consider Schrödinger operators in $\ell^2(\mathbb{Z})$,

$$(1) \quad [H_\omega \psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n),$$

where the potentials are of the form $V_\omega(n) = f(T^n \omega)$, with ω in a compact metric space Ω , a homeomorphism $T : \Omega \rightarrow \Omega$, and $f \in C(\Omega, \mathbb{R})$. We also fix an ergodic measure μ .

Notice that the Hilbert space in which H_ω acts is now ω -independent, and the aperiodic order features of the medium that is being modeled are completely subsumed in the potential V_ω of the operator H_ω . In this we follow the standard convention, for this class of operators has been commonly studied. One could consider operators that are formally more akin to the operators considered above in general dimensions. However, this would not lead to any significant mathematical difference. Loosely speaking, the aperiodically-ordered Delone set in \mathbb{R} is just being reconfigured as \mathbb{Z} , and the local properties of the operator that depend on the pattern near a point in the general setting affect the value of the potential at the point in question accordingly in our present setting.

In fact, this scenario is more general than considered in the previous section. The better analog would be the case where $(T\omega)_n = \omega_{n+1}$ is the shift on $\mathcal{A}^{\mathbb{Z}}$ for some finite set \mathcal{A} , $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ is T -invariant and closed, and $f : \Omega \rightarrow \mathbb{R}$ is locally constant, that is, it only depends on the values of ω_n for some finite set of n values. However, some of the results below hold in the more general setting, and we will impose further restrictions when they are needed.

Consequently, Theorems 3.1 and 3.2 now take the following form.

Theorem 4.1. *There are sets Σ , Σ_{ac} , Σ_{sc} , and Σ_{pp} such that for μ -almost every $\omega \in \Omega$, we have $\sigma(H_\omega) = \Sigma$, $\sigma_{ac}(H_\omega) = \Sigma_{ac}$, $\sigma_{sc}(H_\omega) = \Sigma_{sc}$, and $\sigma_{pp}(H_\omega) = \Sigma_{pp}$.*

Theorem 4.2. *The measures $\int \varphi dN_k^\omega = \frac{1}{k} \text{Tr}(\varphi(H_\omega|_{[1,k]}))$ converge weakly to the measure $\int \varphi dN = \int \langle \delta_0, \varphi(H_\omega) \delta_0 \rangle d\mu(\omega)$ as $k \rightarrow \infty$.*

The second result uses a weaker notion of convergence than in Theorem 3.2, the price we have to pay for casting this problem in the more general setting. However, this is fine after all, due to the following result.

Theorem 4.3. *The measure dN is continuous.*

Moreover, we have:

Theorem 4.4. *The topological support of the measure dN is equal to Σ .*

Theorems 4.1–4.4 are standard results from the theory of ergodic Schrödinger operators on $\ell^2(\mathbb{Z})$; compare [30].

4.1. Spectrum and the Absence of Uniform Hyperbolicity. Let us consider solutions to the difference equation

$$(2) \quad u(n+1) + u(n-1) + V_\omega(n)u(n) = Eu(n), \quad n \in \mathbb{Z},$$

for some energy $E \in \mathbb{R}$. Clearly, u solves (2) if and only if

$$(3) \quad \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \begin{pmatrix} E - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Since $V_\omega(n) = f(T^n\omega)$, one naturally defines

$$A_E(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix},$$

so that (3) implies

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = A_E(T^n\omega) \times \cdots \times A_E(T\omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

for $n \geq 1$ and solutions u to (2). We set $M_{E,\omega}(n) = A_E(T^n\omega) \times \cdots \times A_E(T\omega)$.

Definition 4.5. *We let*

$$\mathcal{UH} = \{E \in \mathbb{R} : \exists c > 1 \text{ such that } \forall \omega \in \Omega, n \geq 1 \text{ we have } \|M_{E,\omega}(n)\| \geq c^n\}.$$

Johnson showed in [97] that the set \mathcal{UH} is equal to the resolvent set of H_ω for any ω that has a dense T -orbit. As a particular consequence, we may state the following:

Theorem 4.6. *Suppose T is minimal. Then for every $\omega \in \Omega$, we have $\sigma(H_\omega) = \mathbb{R} \setminus \mathcal{UH}$. In particular, for any ergodic measure μ , we have $\Sigma = \mathbb{R} \setminus \mathcal{UH}$.*

4.2. Kotani Theory. In the previous subsection, we saw that the partition $\mathbb{R} = \mathcal{UH} \sqcup \mathbb{R} \setminus \mathcal{UH}$ yields the partition of the energy axis into resolvent set and spectrum. Let us partition the energy axis even further by introducing the Lyapunov exponent

$$L_\mu(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|M_{E,\omega}(n)\| d\mu(\omega).$$

The existence of the limit follows quickly by subadditivity. Moreover, due to ergodicity of μ and Kingman's Subadditive Ergodic Theorem, we have

$$L_\mu(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|M_{E,\omega}(n)\| \quad \text{for } \mu\text{-a.e. } \omega \in \Omega.$$

Obviously, we have $L_\mu(E) > 0$ for every $E \in \mathcal{UH}$. We let

$$\mathcal{Z}_\mu = \{E : L_\mu(E) = 0\}$$

and

$$\mathcal{NUH}_\mu = \{E \in \mathbb{R} : L_\mu(E) > 0\} \setminus \mathcal{UH},$$

so that our final partition is $\mathbb{R} = \mathcal{UH} \sqcup \mathcal{NUH}_\mu \sqcup \mathcal{Z}_\mu$. Notice that in these definitions, the dependence of the partition of $\mathbb{R} \setminus \mathcal{UH}$ into $\mathcal{NUH}_\mu \sqcup \mathcal{Z}_\mu$ on the choice of the ergodic measure μ is emphasized through the subscript for the μ -dependent sets. It is customary to drop this explicit subscript from L , \mathcal{Z} , and \mathcal{NUH} , and we will henceforth do so as well.

Recall that the essential closure of a set $S \subseteq \mathbb{R}$ is given by

$$\overline{S}^{\text{ess}} = \{E \in \mathbb{R} : \forall \varepsilon > 0 \text{ we have } \text{Leb}(S \cap (E - \varepsilon, E + \varepsilon)) > 0\}.$$

Then, the following result is the celebrated Ishii-Pastur-Kotani theorem; see, for example, [45, 94, 106, 128, 142].

Theorem 4.7. *We have $\Sigma_{\text{ac}} = \overline{\mathcal{Z}}^{\text{ess}}$.*

Since the essential closure of a set S is empty if and only if S has zero Lebesgue measure, this result yields a characterization of almost sure purely singular spectrum. In fact, this is the typical situation for our models of interest [107].

Theorem 4.8. *Suppose the range of $f : \Omega \rightarrow \mathbb{R}$ is finite and $\text{Leb}(\mathcal{Z}) > 0$. Denote the push-forward of μ under $\Omega \ni \omega \mapsto V_\omega \in (\text{Ran } f)^\mathbb{Z}$ by μ^* . Then, $\text{supp } \mu^*$ is finite.*

Since $\text{supp } \mu$ is T -invariant, this means that every element of $\text{supp } \mu^*$ is periodic. In other words, if the potentials are ergodic, aperiodic and take finitely many values, then \mathcal{Z} has zero Lebesgue measure and the almost sure absolutely continuous spectrum is empty.

4.3. Zero-Measure Spectrum. The realization that $\text{Leb}(\mathcal{Z}) = 0$ for potentials that are ergodic, aperiodic and take finitely many values is at the heart of proofs of zero-measure spectrum in these cases. Whenever Theorem 4.8 applies, it suffices to show that the spectrum is contained in (and hence coincides with) \mathcal{Z} in order to establish zero-measure spectrum. There are two approaches that establish this identity. One shows that the Lyapunov exponent must vanish for every energy in the spectrum. The other is based on Theorem 4.6, which says that $\Sigma = \mathcal{Z} \cup \mathcal{NUH}$. Hence, it suffices to prove that if E is such that the Lyapunov exponent is positive, the convergence of $\frac{1}{n} \log \|M_{E,\omega}(n)\|$ to $L(E)$ must be uniform. The second approach can be made to work in greater generality, whereas the first approach often gives additional information that has other interesting consequences.

We will discuss how the first approach is implemented when we discuss Sturmian models and, specifically, the Fibonacci Hamiltonian in later sections. Here we therefore discuss in some detail how the second approach works. We first state the main result, then explain how it is a natural consequence of the line of reasoning employed. We emphasize that this is a result that holds in the symbolic setting. That is, for a finite set \mathcal{A} , called the *alphabet*, we consider the shift transformation $T : \mathcal{A}^{\mathbb{Z}} \rightarrow \mathcal{A}^{\mathbb{Z}}$ given by $(T\omega)(n) = \omega(n+1)$ and closed, T -invariant subsets Ω of $\mathcal{A}^{\mathbb{Z}}$, which are called *subshifts (over \mathcal{A})*. We denote by \mathcal{W}_Ω the set of all finite words over the alphabet \mathcal{A} that occur in elements of Ω , and we write $\mathcal{W}_\Omega(n)$ for those elements of \mathcal{W}_Ω that have length n . A function on Ω is called *locally constant* if $f(\omega)$ only depends on finitely many entries. More formally, there exists $k \in \mathbb{Z}_+$ such that $f(\omega) = f(\omega')$ for all $\omega, \omega' \in \Omega$ with $\omega_{-k} \dots \omega_k = \omega'_{-k} \dots \omega'_k$.

As usual, a subshift Ω is called *minimal* if the topological dynamical system (Ω, T) is minimal (i.e., the T -orbit of every $\omega \in \Omega$ is dense in Ω), it is called *uniquely ergodic* if (Ω, T) has a unique invariant Borel probability measure, and it is called *strictly ergodic* if it is both minimal and uniquely ergodic. In the uniquely ergodic case, the unique invariant Borel probability measure must necessarily be ergodic.

Definition 4.9. *Let Ω be a strictly ergodic subshift with unique T -invariant measure μ . It satisfies the Boshernitzan condition (B) if*

$$\limsup_{n \rightarrow \infty} \left(\min_{w \in \mathcal{W}_\Omega(n)} n \cdot \mu([w]) \right) > 0.$$

Here is the result from [56] showing that (B) is a sufficient condition for zero-measure spectrum:

Theorem 4.10. *Suppose Ω is a strictly ergodic subshift that satisfies condition (B) and $f : \Omega \rightarrow \mathbb{R}$ is locally constant. Then the convergence of $\frac{1}{n} \log \|M_{E,\omega}(n)\|$ to $L(E)$ is uniform for every $E \in \mathbb{R}$. In particular, \mathcal{NUH} is empty and $\Sigma = \mathcal{Z}$. Thus, if Ω and f are such that the V_ω are aperiodic, then $\text{Leb}(\Sigma) = 0$. In the latter case, $\Sigma_{\text{ac}} = \emptyset$.*

The proof of Theorem 4.10 is more easily understood if one imposes a stronger assumption. Indeed, let us assume for the time being that $f(\omega)$ depends only on ω_0 and $\min_{w \in \mathcal{W}_\Omega(n)} n \cdot \mu([w])$ is uniformly large for all n , not merely for a subsequence. That is, suppose there is $\delta > 0$ such that $|w| \cdot \mu([w]) \geq \delta$ for every $w \in \mathcal{W}_\Omega$.

By our stronger assumption on f , we may view $\log \|M_{E,\omega}(n)\|$ as a quantity associated with the word $w = \omega_1 \dots \omega_n \in \mathcal{W}_\Omega(n)$ that we will denote by $F(w)$. If we do so, then the goal is to prove that $|w|^{-1} F(w)$ converges uniformly as $|w| \rightarrow \infty$ and each w belongs to \mathcal{W}_Ω . It is a known consequence of unique ergodicity of (Ω, T) that the convergence in

$$F^+ := \limsup_{w \in \mathcal{W}_\Omega, |w| \rightarrow \infty} |w|^{-1} F(w)$$

is uniform. By the uniform upper bound and the frequent occurrence of any word due to the strengthened assumption, one can derive a similar uniform result for the \liminf , and hence establish uniform convergence. If one only has condition (B), then a similar way of reasoning can be carried out for the sequence of length scales from (B). In this way, one can establish uniform convergence along this sequence. To interpolate, one can employ the Avalanche Principle from [80].

The paper [57] contains numerous applications of Theorem 4.10 to specific classes of subshifts, some of which will be described in Subsection 4.4 below.

When [56, 57] appeared, all known zero-measure results for Schrödinger operators defined by strictly ergodic subshifts (see, e.g., [9, 11, 13, 25, 109, 110, 117, 146]) were covered by this approach, that is, they all held for subshifts that satisfy condition (B). Recently, Liu and Qu constructed examples of strictly ergodic subshifts that do not satisfy (B) but for which the associated Schrödinger operators do have zero-measure spectrum (and in fact the convergence of $\frac{1}{n} \log \|M_{E,\omega}(n)\|$ to $L(E)$ is uniform for every $E \in \mathbb{R}$) [116].

Naturally, once one knows that the spectrum has zero Lebesgue measure, one would like to determine its fractal (e.g., Hausdorff, lower and upper box counting) dimensions, as well as similar quantities such as thickness and denseness. These more delicate questions have been studied for a rather small number of examples, which will be discussed in subsequent sections. Zero-measure spectrum, on the other hand, is known in much greater generality, and this is the topic of the next subsection.

4.4. Examples. In this subsection, we present several classes of popular examples of potentials that are ergodic, aperiodic, and take finitely many values (so that Kotani's central result applies) and discuss the validity of condition (B) (so that the associated Schrödinger operators have zero-measure spectrum). For more details, we refer the reader to [57].

4.4.1. Linearly Recurrent Subshifts and Subshifts Generated by Primitive Substitutions. A subshift Ω over \mathcal{A} is called *linearly recurrent* (or *linearly repetitive*) if there exists a constant K such that if $v, w \in \mathcal{W}(\Omega)$ with $|w| \geq K|v|$, then v is a subword of w . Clearly, every linearly recurrent subshift Ω satisfies (B). A popular way to generate linearly recurrent subshifts is via primitive substitutions. A substitution $S : \mathcal{A} \rightarrow \mathcal{A}^*$ is called *primitive* if there exists $k \in \mathbb{N}$ such that for every $a, b \in \mathcal{A}$, $S^k(a)$ contains b . Such a substitution generates a subshift Ω as follows. It is easy to see that there are $m \in \mathbb{N}$ and $a \in \mathcal{A}$ such that $S^m(a)$ begins with a . If we iterate S^m on the symbol a , we obtain a one-sided infinite limit, u , called a *substitution sequence*. Ω then consists of all two-sided sequences for which all subwords are also subwords of u . One can verify that this construction is in fact independent of the choice of u , and hence Ω is uniquely determined by S . Prominent examples² are given by

$a \mapsto ab, b \mapsto a$	Fibonacci
$a \mapsto ab, b \mapsto ba$	Thue-Morse
$a \mapsto ab, b \mapsto aa$	Period Doubling
$a \mapsto ab, b \mapsto ac, c \mapsto db, d \mapsto dc$	Rudin-Shapiro

The following was shown in [72] (and independently in [66]):

Proposition 4.11. *If the subshift Ω is generated by a primitive substitution, then it is linearly recurrent and hence satisfies condition (B).*

It may happen that a non-primitive substitution generates a linearly recurrent subshift. An example is given by $a \mapsto aaba, b \mapsto b$. In fact, the class of linearly

²These examples appear explicitly in many papers in the physics literature on Schrödinger operators generated by primitive substitution; compare [2, 6, 14, 26, 78, 90, 105, 119].

recurrent subshifts generated by substitutions was characterized in [58].³ In particular, it turns out that a subshift generated by a substitution is linearly recurrent if and only if it is minimal.

4.4.2. Sturmian and Quasi-Sturmian Subshifts. Consider a minimal subshift Ω over \mathcal{A} . The (factor) complexity function $p : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ is defined by $p(n) = \#\mathcal{W}_\Omega(n)$. Hedlund and Morse showed in [84] that Ω is aperiodic if and only if $p(n) \geq n + 1$ for every $n \in \mathbb{Z}_+$. Aperiodic minimal subshifts of minimal complexity, $p(n) = n + 1$ for every $n \in \mathbb{N}$, exist and they are called Sturmian. If the complexity function satisfies $p(n) = n + k$ for $n \geq n_0$, $k, n_0 \in \mathbb{N}$, the subshift is called quasi-Sturmian. It is known that quasi-Sturmian subshifts are exactly those subshifts that are a morphic image of a Sturmian subshift; compare [32, 34, 132].

There are a large number of equivalent characterizations of Sturmian subshifts; compare [17]. We are mainly interested in their geometric description in terms of an irrational rotation. Let $\alpha \in (0, 1)$ be irrational and consider the rotation by α on the circle, $R_\alpha : [0, 1) \rightarrow [0, 1)$, $R_\alpha \theta = \{\theta + \alpha\}$, where $\{x\}$ denotes the fractional part of x , $\{x\} = x \bmod 1$. The coding of the rotation R_α according to a partition of the circle into two half-open intervals of length α and $1 - \alpha$, respectively, is given by the sequences $v_n(\alpha, \theta) = \chi_{[0, \alpha)}(R_\alpha^n \theta)$. We obtain a subshift

$$\Omega_\alpha = \overline{\{v(\alpha, \theta) : \theta \in [0, 1)\}} = \{v(\alpha, \theta) : \theta \in [0, 1)\} \cup \{\tilde{v}^{(k)}(\alpha) : k \in \mathbb{Z}\} \subset \{0, 1\}^{\mathbb{Z}},$$

which can be shown to be Sturmian. Here, $\tilde{v}_n^{(k)}(\alpha) = \chi_{(0, \alpha]}(R_\alpha^{n+k} 0)$. Conversely, every Sturmian subshift is essentially of this form, that is, if Ω is minimal and has complexity function $p(n) = n + 1$, then, up to a one-to-one morphism, $\Omega = \Omega_\alpha$ for some irrational $\alpha \in (0, 1)$.

Using this description and some classical results in diophantine approximation, the following result was shown in [57].

Theorem 4.12. *Every Sturmian subshift obeys the Boshernitzan condition (B).*

Moreover, establishing stability of (B) under morphic images, one obtains the following consequence.

Corollary 4.13. *Every quasi-Sturmian subshift obeys (B).*

4.4.3. Circle Maps. Let $\alpha \in (0, 1)$ be irrational and $\beta \in (0, 1)$ arbitrary. The coding of the rotation R_α according to a partition into two half-open intervals of length β and $1 - \beta$, respectively, is given by the sequences $v_n(\alpha, \beta, \theta) = \chi_{[0, \beta)}(R_\alpha^n \theta)$. We obtain a subshift

$$\Omega_{\alpha, \beta} = \overline{\{v(\alpha, \beta, \theta) : \theta \in [0, 1)\}} \subset \{0, 1\}^{\mathbb{Z}}.$$

Subshifts generated this way are usually called circle map subshifts or subshifts generated by the coding of a rotation.

The paper [57] established the following results for circle map subshifts in connection with property (B):

Theorem 4.14. *Let $\alpha \in (0, 1)$ be irrational. Then the subshift $\Omega_{\alpha, \beta}$ satisfies (B) for Lebesgue almost every $\beta \in (0, 1)$.*

³See also [69, 70, 115] for results for Schrödinger operators arising from a specific class of non-primitive substitutions.

Theorem 4.15. *Let $\alpha \in (0, 1)$ be irrational with bounded continued fraction coefficients, that is, $a_n \leq C$. Then $\Omega_{\alpha, \beta}$ satisfies (B) for every $\beta \in (0, 1)$.*

Theorem 4.16. *Let $\alpha \in (0, 1)$ be irrational with unbounded continued fraction coefficients. Then there exists $\beta \in (0, 1)$ such that $\Omega_{\alpha, \beta}$ does not satisfy (B).*

4.4.4. Interval Exchange Transformations. Subshifts generated by interval exchange transformations (IETs) are natural generalizations of Sturmian subshifts. IETs are defined as follows. Given a probability vector $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda_i > 0$ for $1 \leq i \leq m$, we let $\mu_0 = 0$, $\mu_i = \sum_{j=1}^i \lambda_j$, and $I_i = [\mu_{i-1}, \mu_i)$. Let τ be a permutation of $\mathcal{A}_m = \{1, \dots, m\}$, that is, $\tau \in S_m$, the symmetric group. Then $\lambda^\tau = (\lambda_{\tau^{-1}(1)}, \dots, \lambda_{\tau^{-1}(m)})$ is also a probability vector, and we can form the corresponding μ_i^τ and I_i^τ . Denote the unit interval $[0, 1)$ by I . The (λ, τ) interval exchange transformation is then defined by

$$T : I \rightarrow I, \quad T(x) = x - \mu_{i-1} + \mu_{\tau(i)-1}^\tau \text{ for } x \in I_i, \quad 1 \leq i \leq m.$$

It exchanges the intervals I_i according to the permutation τ .

The transformation T is invertible, and its inverse is given by the $(\lambda^\tau, \tau^{-1})$ interval exchange transformation.

The symbolic coding of $x \in I$ is $\omega_n(x) = i$ if $T^n(x) \in I_i$. This induces a subshift over the alphabet \mathcal{A}_m : $\Omega_{\lambda, \tau} = \{\omega(x) : x \in I\}$.

Sturmian subshifts correspond to the case of two intervals, as a first return map construction shows.

Keane [100] proved that if the orbits of the discontinuities μ_i of T are all infinite and pairwise distinct, then T is minimal. In this case, the coding is one-to-one and the subshift is minimal and aperiodic. This holds in particular if τ is irreducible and λ is irrational. Here, τ is called irreducible if $\tau(\{1, \dots, k\}) \neq \{1, \dots, k\}$ for every $k < m$ and λ is called irrational if the λ_i are rationally independent.

Regarding property (B), Boshernitzan has proved two results. First, in [20] the following is shown:

Theorem 4.17. *For every irreducible $\tau \in S_m$ and for Lebesgue almost every λ , the subshift $\Omega_{\lambda, \tau}$ satisfies (B).*

In fact, Boshernitzan shows that for every irreducible $\tau \in S_m$ and for Lebesgue almost every λ , the subshift $\Omega_{\lambda, \tau}$ satisfies a stronger condition where the sequence of n values for which $\eta(n)$ is large cannot be too sparse. This condition is easily seen to imply (B), and hence the theorem above.

In a different paper, [21], Boshernitzan singles out an explicit class of subshifts arising from interval exchange transformations that satisfy (B). The transformation T is said to be of (rational) rank k if the μ_i span a k -dimensional space over \mathbb{Q} (the field of rational numbers).

Theorem 4.18. *If T has rank 2, the subshift $\Omega_{\lambda, \tau}$ satisfies (B).*

4.5. Singular Continuous Spectrum. As seen in the previous section, the spectrum has zero Lebesgue measure whenever condition (B) holds. This condition is satisfied by a wide class of models, in particular by all typical quasicrystal models. As pointed out in Theorem 4.10, a consequence of zero-measure spectrum is the absence of absolutely continuous spectrum. That is, if $\sigma(H_\omega)$ has zero Lebesgue

measure, then $\sigma_{\text{ac}}(H_\omega) = \emptyset$, since all spectral measures are supported by the spectrum, and any measure supported by a set of zero Lebesgue measure must be purely singular by definition.

To complement this, one can often show the absence of point spectrum. That is, there are a variety of tools that allow one to show that H_ω has no eigenvalues, and hence $\sigma_{\text{pp}}(H_\omega) = \emptyset$ as well. Putting the two results together, one obtains that H_ω has purely singular continuous spectrum.

The primary tool that allows one to exclude eigenvalues is based on the Gordon lemma, which assumes that the potential has infinitely many suitably aligned local periodicities. Overall, this nicely implements the philosophy that aperiodic order is intermediate between periodic and random. The aperiodicity implies the absence of absolutely continuous spectrum via Kotani's theorem (and hence one does not have the spectral type that appears for a periodic medium), while the order feature implies the absence of point spectrum via a fingerprint of local periodicity (and hence one does not have the spectral type that appears for a random medium).

A potential $V : \mathbb{Z} \rightarrow \mathbb{R}$ is called a *Gordon potential* if there are $q_k \rightarrow \infty$ such that for every k , we have $V(n) = V(n + q_k) = V(n - q_k)$ for $1 \leq n \leq q_k$. That is, V looks like a periodic potential around the origin, as one sees at least three suitably aligned periodic unit cells there, and the period may be chosen arbitrarily large. The following Gordon lemma is based in spirit on [82]. In this particular form it was shown in [67].

Lemma 4.19. *Suppose V is a Gordon potential. Then, for every E , the difference equation*

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

has no non-trivial square-summable solutions. In particular, the associated Schrödinger operator H in $\ell^2(\mathbb{Z})$, given by

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n),$$

has empty point spectrum.

By ergodicity, T -invariance, and the Gordon lemma, if one can show that

$$\mu(\{\omega \in \Omega : V_\omega \text{ is a Gordon potential}\}) > 0,$$

then

$$\mu(\{\omega \in \Omega : H_\omega \text{ has empty point spectrum}\}) = 1.$$

On the other hand, for any aperiodic minimal subshift, at least one of its elements fails to have the required Gordon three-block symmetries [40]. Thus, one cannot use this approach to show uniform absence of eigenvalues for all $\omega \in \Omega$. Nevertheless, results to this effect are known, established with the following variant of the Gordon lemma.

Lemma 4.20. *Suppose $V : \mathbb{Z} \rightarrow \mathbb{R}$ is such that there are $q_k \rightarrow \infty$ such that $V(n) = V(n + q_k)$ for $1 \leq n \leq q_k$. Suppose further that E is such that*

$$(4) \quad \sup_k \left| \text{Tr} \left(\begin{pmatrix} E - V(q_k) & -1 \\ 1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} E - V(1) & -1 \\ 1 & 0 \end{pmatrix} \right) \right| < \infty.$$

Then, the difference equation

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

has no non-trivial solutions that are square-summable on \mathbb{Z}_+ and hence E is not an eigenvalue of the associated Schrödinger operator H in $\ell^2(\mathbb{Z})$. In particular, if the assumption (4) holds for every E in the spectrum of H , then H has empty point spectrum.

In many quasicrystal models, the existence of hierarchical structures gives rise to a so-called trace map, which in turn can often be used to ensure that (4) holds for all energies in the spectrum. Thus, the analysis then reduces to finding suitable squares of arbitrary length starting at the origin.

Thus, in the symbolic setting at hand, the observations above give rise to problems that concern the subword structure of the potentials, and hence fall in the general area of combinatorics on words.

Let us describe the results that have been obtained in this way for the examples discussed above. In all these results, the choice of the sampling function f is more restricted than above. Namely, one usually assumes that $f(\omega) = g(\omega_0)$ with an injective map $g : \mathcal{A} \rightarrow \mathbb{R}$.

4.5.1. Subshifts Generated by Primitive Substitutions. Suppose S is a primitive substitution over the alphabet \mathcal{A} and let $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ be the subshift associated with it. Recall that it is strictly ergodic and denote the unique invariant probability measure by μ . The *index* of Ω is given by the largest fractional power occurring in an (and hence any) element of Ω . Formally, the index is defined as follows. Given $w \in \mathcal{W}_\Omega$ and any prefix v of w , we denote the word $w^k v$ with $k \in \mathbb{Z}_+$ by w^r , where $r = k + \frac{|v|}{|w|}$. Then, $\text{ind}_\Omega(w) = \sup\{r \in \mathbb{Q} \cap [1, \infty) : w^r \in \mathcal{W}_\Omega\}$ and $\text{ind}(\Omega) = \sup\{\text{ind}_\Omega(w) : w \in \mathcal{W}_\Omega\}$.

The following result was shown in [40].⁴

Theorem 4.21. *If S is a primitive substitution and the associated subshift Ω satisfies $\text{ind}(\Omega) > 3$, then $\mu(\{\omega \in \Omega : V_\omega \text{ is a Gordon potential}\}) > 0$. Consequently, $\mu(\{\omega \in \Omega : H_\omega \text{ has empty point spectrum}\}) = 1$.*

The underlying idea is simple. Since the subshift is invariant under S , any word appearing with index strictly greater than 3 generates by iteration of S a sequence of words whose lengths go to infinity and whose index is bounded away from 3. This allows one to bound from below the frequency with which third powers occur and hence yields measure estimates on the Gordon three-block conditions that are good enough to show that the limsup of these sets must have positive measure. Since the elements of the limsup of these sets give rise to Gordon potentials, the result follows.

Applications of this theorem include the Fibonacci substitution (since $\text{ind}(\Omega) \geq \text{ind}_\Omega(abaab) \geq 3.2$), the period doubling substitution (since $\text{ind}(\Omega) \geq \text{ind}_\Omega(ab) \geq 3.5$), and many others. Of course, the result does not apply to the Thue-Morse substitution, which is famous mainly because $\text{ind}(\Omega) = 2$. Unfortunately, it is still open whether the point spectrum is almost surely empty in the Thue-Morse case. The Gordon approach fails due to small index, and no other methods are known that yield an almost sure result.⁵

⁴See [39] for a precursor dealing with the period doubling case. In this special case it was later shown that the absence of eigenvalues even holds for all $\omega \in \Omega$ [42].

⁵The absence of eigenvalues for a dense G_δ set of ω 's can be established in this example and many others using palindromes instead of powers [89]. However, using palindromes one cannot prove the absence of eigenvalues for a full measure set of ω 's [66].

4.5.2. Sturmian and Quasi-Sturmian Subshifts. Damanik, Killip, and Lenz showed the following result in [52] (see also [53] for a uniform result for almost every Sturmian subshift).

Theorem 4.22. *For every Sturmian subshift Ω , H_ω has empty point spectrum for every $\omega \in \Omega$.*

This result was the culmination of a sequence of partial results. Among those, we single out Sütő [145], who proved empty point spectrum for one α and one ω , Bellissard et al. [13], who proved it for all α and one ω , Delyon-Petritis [67], who proved it for almost all α and almost all ω , and Kaminaga [98], who proved it for all α and almost all ω . Here, $\alpha \in (0, 1) \setminus \mathbb{Q}$ denotes the slope associated with a Sturmian subshift. Recall that Sturmian subshifts are in one-to-one correspondence with $(0, 1) \setminus \mathbb{Q}$. Here, [13, 52, 53, 145] used Lemma 4.20, whereas [67, 98] used Lemma 4.19.

The extension of Theorem 4.22 to the quasi-Sturmian case was obtained by Damanik and Lenz in [55].

Theorem 4.23. *For every quasi-Sturmian subshift Ω , H_ω has empty point spectrum for every $\omega \in \Omega$.*

4.5.3. Circle Maps. Recall that a circle map subshift is determined by the parameters $\alpha \in (0, 1) \setminus \mathbb{Q}$ and $\beta \in (0, 1)$. It is strictly ergodic and we denote the unique ergodic measure by μ . Delyon and Petritis proved the following in [67].

Theorem 4.24. *For almost every α and every β , the corresponding circle map subshift Ω is such that $\mu(\{\omega \in \Omega : V_\omega \text{ is a Gordon potential}\}) = 1$. Consequently, $\mu(\{\omega \in \Omega : H_\omega \text{ has empty point spectrum}\}) = 1$.*

In fact, the full measure set of α values is explicitly described in terms of the continued fraction expansion. The condition was weakened by Kaminaga in [98], still however excluding an explicit zero measure set. This weaker condition was only shown to imply $\mu(\{\omega \in \Omega : V_\omega \text{ is a Gordon potential}\}) > 0$, which of course is still sufficient to allow one to deduce $\mu(\{\omega \in \Omega : H_\omega \text{ has empty point spectrum}\}) = 1$.

4.5.4. Interval Exchange Transformations. Recall that an IET subshift is determined by an irreducible permutation τ and a probability vector λ . Cobo et al. showed the following result in [33] (see also [68]).

Theorem 4.25. *For every irreducible permutation τ and almost every λ , the associated IET subshift Ω is such that $\mu(\{\omega \in \Omega : V_\omega \text{ is a Gordon potential}\}) = 1$. Consequently, $\mu(\{\omega \in \Omega : H_\omega \text{ has empty point spectrum}\}) = 1$.*

4.6. Transport Properties. Quasicrystal models have behavior that is markedly different from the periodic and random cases in many different respects. In the previous subsections we have seen that the spectrum is typically a zero-measure Cantor set, while for periodic and random potentials it is always given by a finite union of non-degenerate compact intervals. Moreover, the spectral type is typically singular continuous, while it is always absolutely continuous in the periodic case and pure point in the (one-dimensional) random case.

In this subsection we consider yet another perspective from which the quasicrystal model behavior is expected to differ from the behavior of the periodic and random cases. Namely, we consider the spreading of wave packets under the time-dependent Schrödinger equation. That is, given a Schrödinger operator H_ω and a

normalized element ψ of $\ell^2(\mathbb{Z})$, we consider $\psi(t) = e^{-itH_\omega}\psi$, where e^{-itH_ω} is defined via the spectral theorem. Then, $\psi(\cdot)$ satisfies the time-dependent Schrödinger equation $i\partial_t\psi(t) = H_\omega\psi(t)$ with initial condition $\psi(0) = \psi$. The quantum mechanical interpretation is that the probability of finding the quantum particle at site $n \in \mathbb{Z}$ at time $t \in \mathbb{R}$ is given by $a(n, t) := |\langle \delta_n, \psi(t) \rangle|^2$. The initial state is naturally localized in some fixed compact set, up to a small error, since it belongs to $\ell^2(\mathbb{Z})$. More specifically, one is often interested in the initial state $\psi = \delta_0$ (or some δ_n), which is completely localized. After having fixed the initial state, one is then interested in how fast $\psi(t)$ spreads out in space, or more specifically, how long one has to wait until $a(n, t)$ is no longer negligibly small at some distant site n . In general, this is a difficult problem. Questions of this kind are easier to study for compound quantities; that is, some averaging in n and/or t helps one generate quantities for which interesting statements can be proven.

A popular way to average in time is to consider Cesàro averages,

$$\tilde{a}(n, T) = \frac{1}{T} \int_0^T a(n, t) dt = \frac{1}{T} \int_0^T |\langle \delta_n, \psi(t) \rangle|^2 dt.$$

Let also

$$M_p(t) = \sum_{n \in \mathbb{Z}} (1 + |n|^p) a(n, t), \quad \tilde{M}_p(T) = \sum_{n \in \mathbb{Z}} (1 + |n|^p) \tilde{a}(n, T), \quad p > 0.$$

Notice that for t (resp., T) fixed, $a(\cdot, t)$ and $\tilde{a}(\cdot, T)$ are probability distributions on \mathbb{Z} , and hence the quantities above are (1 plus) the p -th moment of the respective probability distribution. Here we assume that the initial state is either a Dirac delta function or at least sufficiently well localized so that these moments are finite.

Wave packet spreading then is reflected by growth in time of these moments. To detect power-law growth, one introduces the so-called transport exponents

$$\begin{aligned} \beta^+(p) &= \limsup_{t \rightarrow \infty} \frac{\log M_p(t)}{p \log t}, & \beta^-(p) &= \liminf_{t \rightarrow \infty} \frac{\log M_p(t)}{p \log t}, \\ \tilde{\beta}^+(p) &= \limsup_{T \rightarrow \infty} \frac{\log \tilde{M}_p(T)}{p \log T}, & \tilde{\beta}^-(p) &= \liminf_{T \rightarrow \infty} \frac{\log \tilde{M}_p(T)}{p \log T}. \end{aligned}$$

Each of these four functions of p is non-decreasing in p and takes values in the interval $[0, 1]$.

In view of the monotonicity of the transport exponents, it is natural to consider their limiting values for small and large values of p . Thus, denote

$$\begin{aligned} \alpha_\ell^\pm &= \lim_{p \downarrow 0} \beta^\pm(p), & \alpha_u^\pm &= \lim_{p \uparrow \infty} \beta^\pm(p), \\ \tilde{\alpha}_\ell^\pm &= \lim_{p \downarrow 0} \tilde{\beta}^\pm(p), & \tilde{\alpha}_u^\pm &= \lim_{p \uparrow \infty} \tilde{\beta}^\pm(p). \end{aligned}$$

We note that there are other useful ways of capturing wave packet spreading, and refer the reader to [65] for a comprehensive survey.

The transport exponents take the constant value 0 for random potentials and (at least the time-averaged quantities) the constant value 1 for periodic potentials. Thus if one is able to prove the occurrence of fractional values of the transport exponents, one exhibits wave packet spreading that is strictly intermediate between the periodic and random cases. Results of this kind are notoriously difficult to establish. The few known results for quasicrystal models will be described in detail in later sections on Fibonacci and Sturmian potentials.

5. THE FIBONACCI HAMILTONIAN

The Fibonacci Hamiltonian is the most prominent model in the study of electronic properties of quasicrystals. It is given by the discrete one-dimensional Schrödinger operator

$$(5) \quad [H_{\lambda,\omega}u](n) = u(n+1) + u(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha + \omega \bmod 1)u(n),$$

where $\lambda > 0$ is the coupling constant, $\alpha = \frac{\sqrt{5}-1}{2}$ is the frequency, and $\omega \in [0, 1)$ is the phase. An alternative way to obtain the same potential is via the Fibonacci substitution; see Section 4.4.1 above. This operator family has been studied in many papers since the early 1980's (see, e.g., [2, 78, 90, 91, 92, 103, 104, 105, 113, 126, 127, 150] for early works in the physics literature), and numerous fundamental results are known. In this section we describe the current “state of the art” for this model.

5.1. Trace Map Formalism. Even the earliest papers on the Fibonacci Hamiltonian realized the importance of a certain renormalization procedure in its study, see [103, 126]. This led in particular to the consideration of a certain dynamical system, the so-called trace map, whose properties are closely related to many spectral properties of the operator (5). The existence of the trace map and its connection to spectral properties is a consequence of the invariance of the potential under a substitution rule. This works in great generality; see [3, 41] and references therein.

The one-step transfer matrices associated with the difference equation $H_{\lambda,\omega}u = Eu$ are given by

$$T_{\lambda,\omega}(m, E) = \begin{pmatrix} E - \lambda\chi_{[1-\alpha,1)}(m\alpha + \omega \bmod 1) & -1 \\ 1 & 0 \end{pmatrix}.$$

Denote the Fibonacci numbers by $\{F_k\}$, that is, $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$. Then the fact that the potential for zero phase is invariant under the Fibonacci substitution implies that the matrices

$$M_{-1}(E) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}, \quad M_0(E) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$M_k(E) = T_{\lambda,0}(F_k, E) \times \cdots \times T_{\lambda,0}(1, E) \quad \text{for } k \geq 1$$

obey the recursive relations

$$M_{k+1}(E) = M_{k-1}(E)M_k(E)$$

for $k \geq 0$. Passing to the variables

$$x_k(E) = \frac{1}{2}\text{Tr}M_k(E),$$

this in turn implies

$$(6) \quad x_{k+1}(E) = 2x_k(E)x_{k-1}(E) - x_{k-2}(E)$$

for $k \geq 1$, with $x_{-1}(E) = 1$, $x_0(E) = E/2$, and $x_1 = (E - \lambda)/2$. The recursion relation (6) exhibits a conserved quantity; namely, we have

$$(7) \quad x_{k+1}(E)^2 + x_k(E)^2 + x_{k-1}(E)^2 - 2x_{k+1}(E)x_k(E)x_{k-1}(E) - 1 = \frac{\lambda^2}{4}$$

for every $k \geq 0$.

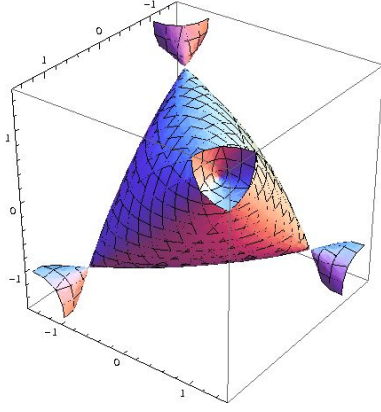


FIGURE 1. The surface $S_{0.01}$.

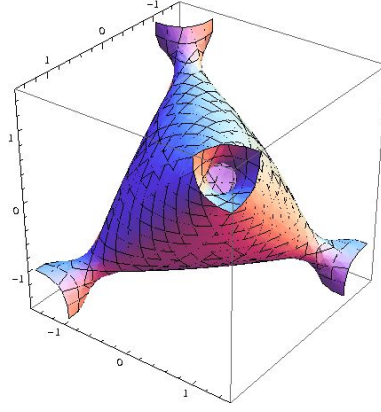


FIGURE 2. The surface $S_{0.5}$.

Given these observations, it is then convenient to introduce the *trace map*

$$(8) \quad T : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad T(x, y, z) = (2xy - z, x, y).$$

Aside from the context described here, this map appears in a natural way in problems related to dynamics of mapping classes [79], Fuchsian groups [18], number theory [19], Painlevé sixth equations [29, 95], the Ising model for quasicrystals [14, 85, 153, 154], the Fibonacci quantum walk [137, 138], and others [7, 59, 148, 155]. See [28] or [16] for an algebraic explanation of this universality. We refer the reader also to [139, 140] for further reading on the Fibonacci trace map.

The following function,

$$G(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1,$$

is invariant under the action of T ,⁶ and hence T preserves the family of cubic surfaces⁷

$$(9) \quad S_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 2xyz = 1 + \frac{\lambda^2}{4} \right\}.$$

Plots of the surfaces $S_{0.01}$ and $S_{0.5}$ are given in Figures 1 and 2, respectively.

Denote by ℓ_λ the line

$$\ell_\lambda = \left\{ \left(\frac{E - \lambda}{2}, \frac{E}{2}, 1 \right) : E \in \mathbb{R} \right\}.$$

It is easy to check that $\ell_\lambda \subset S_\lambda$.

Sütő proved the following central result in [145].

Theorem 5.1. *An energy E belongs to the spectrum of $H_{\lambda, \omega}$ if and only if the positive semiorbit of the point $(\frac{E - \lambda}{2}, \frac{E}{2}, 1)$ under iterates of the trace map T is bounded.*

⁶The function $G(x, y, z)$ is usually called the *Fricke character* or *Fricke-Vogt invariant*.

⁷The surface S_0 is called the *Cayley cubic*.

To obtain this theorem, Sütő argued as follows. Denote

$$\sigma_k = \{E \in \mathbb{R} : |x_k(E)| \leq 1\}$$

and

$$\Sigma_k = \sigma_k \cup \sigma_{k+1}.$$

These sets depend on the coupling constant λ , and whenever we want to make this dependence explicit, we will write $\sigma_{k,\lambda}$ and $\Sigma_{k,\lambda}$. An analysis of the trace recursion (6) shows that the sets Σ_k are decreasing, and hence it is natural to consider their limit $\tilde{\Sigma} = \bigcap \Sigma_k$. Clearly, if $E \in \tilde{\Sigma}$, then $\{x_n(E)\}$ remains bounded due to (7). On the other hand, the analysis of the trace recursion (6) also yields that whenever $E \notin \Sigma_k$ for some k , then $|x_{n-k}(E)|$ obeys an explicit super-exponentially growing lower bound. That is, the sequence $\{x_n(E)\}$ remains bounded if and only if $E \in \tilde{\Sigma}$. Notice that the point $(\frac{E-\lambda}{2}, \frac{E}{2}, 1)$ is just $(x_1(E), x_0(E), x_{-1}(E))$, so that Theorem 5.1 follows as soon as $\Sigma = \tilde{\Sigma}$ is established. The inclusion $\Sigma \subseteq \tilde{\Sigma}$ holds since σ_k is precisely the spectrum of the canonical periodic approximant of period F_k and the fact that these periodic approximants converge strongly. The inclusion $\tilde{\Sigma} \subseteq \Sigma$ holds since one can use the boundedness of $\{x_n(E)\}$ for $E \in \tilde{\Sigma}$ along with the Gordon lemma to show that no solution for this energy is square-summable at $+\infty$, which implies that E must be in the spectrum.

5.2. Hyperbolicity of the Trace Map. Let $f : M \rightarrow M$ be a diffeomorphism of a Riemannian manifold M . Let us recall that an invariant closed set Λ of the diffeomorphism f is *hyperbolic* if there exists a splitting of a tangent space $T_x M = E_x^s \oplus E_x^u$ at every point $x \in \Lambda$ such that this splitting is invariant under Df , and the differential Df exponentially contracts vectors from stable subspaces $\{E_x^s\}$ and exponentially expands vectors from unstable subspaces $\{E_x^u\}$. A hyperbolic set Λ of a diffeomorphism $f : M \rightarrow M$ is *locally maximal* if there exists a neighborhood $U(\Lambda)$ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

We will consider diffeomorphisms of a surface, $\dim M = 2$, and hyperbolic sets of topological dimension zero. In this case a locally maximal hyperbolic set Λ can be locally represented as a product of “stable” and “unstable” Cantor sets C^s and C^u . Both Cantor sets C^s and C^u are *dynamically defined* in the sense of the following definition.

Definition 5.2. Let $I \subset \mathbb{R}^1$ be a closed interval. A Cantor set $C \subset I$ is *dynamically defined* if there are strictly monotone contracting maps $\psi_1, \psi_2, \dots, \psi_k : I \rightarrow I$, $\psi_i(I) \cap \psi_j(I) = \emptyset$ if $i \neq j$, such that $C = \bigcap_{n \in \mathbb{N}} I_n$, where $I_1 = \psi_1(I) \cup \dots \cup \psi_k(I)$ and $I_{n+1} = \psi_1(I_n) \cup \dots \cup \psi_k(I_n)$.

If $\psi_1, \psi_2, \dots, \psi_k$ are $C^{1+\varepsilon}$ -functions, then the Cantor set has zero measure, depends continuously on ψ_1, \dots, ψ_k , and is “regular” in many other ways. We will be interested in the *Hausdorff dimension* and the *thickness* of the Cantor sets C^s and C^u . Denote the Hausdorff dimension of the set C by $\dim_H C$.

In our case, $\dim_H \Lambda = \dim_H C^s + \dim_H C^u$; see [122, 130]. Moreover, if $f \in C^r$ depends smoothly on a parameter, then $\dim_H \Lambda$ is also a smooth function of the parameter; see [120].

Definition 5.3. Let $C \subset \mathbb{R}$ now be an arbitrary Cantor set and denote by I its convex hull. Any connected component of $I \setminus C$ is called a gap of C . A presentation of C is given by an ordering $\mathcal{U} = \{U_n\}_{n \geq 1}$ of the gaps of C . If $u \in C$ is a boundary point of a gap U of C , we denote by K the connected component of $I \setminus (U_1 \cup U_2 \cup \dots \cup U_n)$ (with n chosen so that $U_n = U$) that contains u and write

$$\tau(C, \mathcal{U}, u) = \frac{|K|}{|U|}.$$

With this notation, the thickness $\tau(C)$ and the denseness $\theta(C)$ of C are given by

$$(10) \quad \tau(C) = \sup_{\mathcal{U}} \inf_u \tau(C, \mathcal{U}, u), \quad \theta(C) = \inf_{\mathcal{U}} \sup_u \tau(C, \mathcal{U}, u).$$

The thickness and the denseness of a Cantor set C are related to the Hausdorff dimension of C by the following inequalities (cf. [129, Section 4.2]),

$$(11) \quad \frac{\log 2}{\log(2 + \frac{1}{\tau(C)})} \leq \dim_H C \leq \frac{\log 2}{\log(2 + \frac{1}{\theta(C)})}.$$

For more details on thickness, see [71, 123, 129]. An important property of thickness was discovered by Newhouse [125]:

Theorem 5.4. If C_1 and C_2 are two Cantor sets and $\tau(C_1) \cdot \tau(C_2) \geq 1$, then the sum $C_1 + C_2$ contains an interval. In the special case $C_1 = C_2 =: C$, we have that $\tau(C) \geq 1$ implies that $C + C$ is an interval.

Consider the restriction $T_\lambda : S_\lambda \rightarrow S_\lambda$ of the trace map T from (8) to the invariant surface S_λ , $T_\lambda = T|_{S_\lambda}$. Denote by Ω_λ the set of points in S_λ whose full orbits under T_λ are bounded.

Theorem 5.5. For every $\lambda > 0$, the set Ω_λ is a locally maximal hyperbolic set of $T_\lambda : S_\lambda \rightarrow S_\lambda$. It is homeomorphic to a Cantor set.

Theorem 5.5 was proved for $\lambda \geq 16$ by Casdagli [31], for small values of λ by Damanik and Gorodetski [47], and finally for all $\lambda > 0$ by Cantat [28].

It is easy to check that $\ell_\lambda \subset S_\lambda$, and therefore the set of points on ℓ_λ whose forward semiorbits are bounded is exactly equal to $\ell_\lambda \cap W^s(\Omega_\lambda)$. Then the spectrum Σ_λ is affine equivalent to the set $\ell_\lambda \cap W^s(\Omega_\lambda)$. It is known that ℓ_λ intersects the leaves of $W^s(\Omega_\lambda)$ transversally for $\lambda \geq 16$ (see Section 2 in [32]), and for sufficiently small $\lambda > 0$ (see [47]). This allows one to consider the spectrum Σ_λ as a dynamically defined Cantor set for these values of the coupling constant. Therefore the following holds.

Corollary 5.6. For $\lambda \geq 16$, as well as for sufficiently small $\lambda > 0$, the spectrum Σ_λ is a dynamically defined Cantor set, and hence:

(i) For every small $\varepsilon > 0$ and every $x \in \sigma(H_{\lambda, \omega})$, we have

$$\begin{aligned} \dim_H((x - \varepsilon, x + \varepsilon) \cap \sigma(H_{\lambda, \omega})) &= \dim_B((x - \varepsilon, x + \varepsilon) \cap \sigma(H_{\lambda, \omega})) \\ &= \dim_H \sigma(H_{\lambda, \omega}) \\ &= \dim_B \sigma(H_{\lambda, \omega}). \end{aligned}$$

(ii) The Hausdorff dimension $\dim_H \sigma(H_{\lambda, \omega})$ is an analytic function of λ , and is strictly between zero and one.

We conjecture that the same holds for all $\lambda > 0$.

5.3. Quantitative Characteristics of the Spectrum at Large Coupling.

The fact that the box counting dimension of the spectrum exists and coincides with its Hausdorff dimension allows one to determine the asymptotic behavior of this λ -dependent quantity in the large coupling limit. In fact, Damanik-Embree-Gorodetski-Tcheremchantsev proved the following in [46].

Theorem 5.7. *We have*

$$\lim_{\lambda \rightarrow \infty} (\dim \Sigma_\lambda) \cdot \log \lambda = \log(1 + \sqrt{2}).$$

Let us briefly explain how this result is obtained. Recall that the spectrum is related to the spectra of the canonical periodic approximants by

$$\Sigma_\lambda = \bigcap_{k \geq 1} \Sigma_{k,\lambda} = \bigcap_{k \geq 1} \sigma_{k,\lambda} \cup \sigma_{k+1,\lambda}.$$

Since each periodic spectrum $\sigma_{k,\lambda}$ is a finite union of non-degenerate compact intervals and the lengths of these intervals can be shown to be decaying, it is natural to use $\Sigma_{k,\lambda}$ as one possible cover of Σ_λ and estimate the Hausdorff dimension of Σ_λ from above in this way. On the other hand, since each interval of $\sigma_{k,\lambda}$ can be shown to have non-empty intersection with Σ_λ , one can estimate the box counting dimension of Σ_λ from below in this way. We observe how crucial it is that these dimensions coincide here. Thus, the analysis of the participating intervals comes down to proving good estimates for their length.

To estimate the length, one makes use of the following basic fact from one-dimensional Floquet theory. The preimage of the *open* interval $(-1, 1)$ under x_k consists of exactly F_k disjoint open intervals, on which x_k is strictly monotone. In fact, in this particular case, the same statement is true for the corresponding *closed* intervals (i.e., the periodic spectra in question have all their gaps open). Thus, the length of one of these intervals (say $I = [a, b]$) can be estimated as follows. Since

$$2 = |x_k(a) - x_k(b)| = \int_a^b |x'_k(E)| dE,$$

we have

$$\frac{2}{\max_{E \in I} |x'_k(E)|} \leq |I| \leq \frac{2}{\min_{E \in I} |x'_k(E)|}.$$

In order to prove estimates for $|x'_k(E)|$, one differentiates the trace recursion (6) and proceeds inductively, making use of the trace invariant (7). This approach was pioneered by Raymond [136] and then used in many subsequent papers. In this inductive approach, it turns out to be important to determine, for a given energy E in one of the intervals of $\sigma_{k,\lambda}$, in how many of the earlier sets $\sigma_{k',\lambda}$, $k' < k$, the energy E in question lies. This gives rise to a combinatorial question that was completely answered in [46]. Combining these combinatorial results with the length estimates one can prove in this way for the intervals in question, the overall strategy above yields the following specific estimates:

$$(12) \quad \dim_H \Sigma_\lambda \leq \frac{\log(1 + \sqrt{2})}{\log \left(\frac{1}{2} \left[(\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12} \right] \right)} \quad \text{for } \lambda \geq 8,$$

$$(13) \quad \dim_B^- \Sigma_\lambda \geq \frac{\log(1 + \sqrt{2})}{\log(2\lambda + 22)} \quad \text{for } \lambda > 4.$$

Theorem 5.7 is then a direct consequence of these estimates and the fact that the Hausdorff dimension and the box counting dimension of Σ_λ are equal for $\lambda \geq 16$.

5.4. Quantitative Characteristics of the Spectrum at Small Coupling.

Fractal properties of Σ_λ for small λ were studied in [49]. Among many other things, that paper established the following pair of theorems.

Theorem 5.8. *We have*

$$\lim_{\lambda \rightarrow 0} \dim \Sigma_\lambda = 1.$$

More precisely, there are constants $C_1, C_2 > 0$ such that

$$1 - C_1\lambda \leq \dim \Sigma_\lambda \leq 1 - C_2\lambda$$

for $\lambda > 0$ sufficiently small.

Theorem 5.9. *We have*

$$\lim_{\lambda \rightarrow 0} \tau(\Sigma_\lambda) = \infty.$$

More precisely, there are constants $C_3, C_4 > 0$ such that

$$C_3\lambda^{-1} \leq \tau(\Sigma_\lambda) \leq \theta(\Sigma_\lambda) \leq C_4\lambda^{-1}$$

for $\lambda > 0$ sufficiently small.

Theorem 5.8 is a consequence of the connection (11) between the Hausdorff dimension of a Cantor set and its denseness and thickness, along with the estimates for the latter quantities provided by Theorem 5.9.

Let us briefly explain how Theorem 5.9 can be obtained. The surface S_0 , from (9), is called the Cayley cubic; it has four conic singularities and can be represented as a union of a two dimensional sphere (with four conic singularities) and four unbounded components. The restriction of the trace map to the sphere is a pseudo-Anosov map (a factor of a hyperbolic map of a two-torus), and its Markov partition can be presented explicitly (see [31] or [47, 49]). For small values of λ , the map $T : S_\lambda \rightarrow S_\lambda$ “inherits” the hyperbolicity of this pseudo-Anosov map everywhere away from the singularities. The dynamics near the singularities must be considered separately. Consider the dynamics of T near one of the singularities, say, near the point $p = (1, 1, 1)$. The set $\text{Per}_2(T)$ of periodic orbits of period two is a smooth curve that contains the point p and intersects S_λ at two points (denote them by $p_1(\lambda)$ and $p_2(\lambda)$) for $\lambda > 0$. Finite pieces of stable and unstable manifolds of $p_1(\lambda)$ and $p_2(\lambda)$ are a distance of order λ from each other. In order to estimate the thickness (and the denseness) of the spectrum Σ_λ , we notice first that the Markov partition for $T : S_0 \rightarrow S_0$ can be continuously extended to a Markov partition for $T : S_\lambda \rightarrow S_\lambda$. The extended Markov partition is formed by finite parts of the stable and unstable manifolds of $p_1(\lambda)$, $p_2(\lambda)$, and the other six periodic points that are continuations of the three remaining singularities. Therefore the size of the elements of these Markov partitions remains bounded, and the size of the distance between them is of order λ . The natural approach now is to use the distortion property (see, e.g., [129]) to show that for the iterated Markov partition, the ratio of the distance between the elements to the size of an element is of the same order. The main technical problem here is again the dynamics of the trace map near the singularities, since the curvature of S_λ is very large there for small λ . Nevertheless, one can still estimate the distortion that is obtained during a transition through a neighborhood of a singularity and prove boundedness of the distortion for arbitrarily large iterates of the trace map. This implies Theorem 5.9.

5.5. The Density of States Measure. Let us now turn to the formulation of results involving the integrated density of states, a quantity of fundamental importance associated with an ergodic family of Schrödinger operators. The integrated density of states (IDS) was introduced in Section 3.2 in a more general context, and represents the distribution function of a density of states measure – a measure supported on the spectrum and, in particular, reflecting the asymptotic distribution of eigenvalues of finite dimensional approximations.

Denote the density of states measure of the Fibonacci Hamiltonian for a given coupling constant λ by dN_λ . Repeating the definition from Section 3.2 in this particular case, we have

$$(14) \quad N_\lambda(E) = \lim_{n \rightarrow \infty} \frac{\#\{\text{eigenvalues of } H_{\lambda,\omega,[1,n]} \text{ that are } \leq E\}}{n},$$

where $H_{\lambda,\omega,[1,n]}$ is the restriction of $H_{\lambda,\omega}$ to the interval $[1, n]$ with Dirichlet boundary conditions, and the limit does not actually depend on the phase ω .

It is interesting to analyze the regularity of the density of states measure. This question was studied for general potentials [35, 36, 37, 77, 114], random potentials [27, 143], and analytic quasi-periodic potentials [5, 22, 23, 24, 80, 81, 83, 141]. In the case of Fibonacci Hamiltonian, the IDS is Hölder continuous.

Theorem 5.10. *For every $\lambda > 0$, there exist $C_\lambda < \infty$ and $\gamma_\lambda > 0$ such that*

$$|N_\lambda(E_1) - N_\lambda(E_2)| \leq C_\lambda |E_1 - E_2|^{\gamma_\lambda}$$

for every E_1, E_2 with $|E_1 - E_2| < 1$.

This follows directly from [52]; see also [38, 54, 92, 93, 96] for some previous related results.

It is also interesting to obtain the asymptotics of the optimal Hölder exponent for large and small couplings.

In the large coupling regime, we have the following [51]:

Theorem 5.11. (a) *Suppose $\lambda > 4$. Then for every*

$$\gamma < \frac{3 \log(\alpha^{-1})}{2 \log(2\lambda + 22)},$$

there is some $\delta > 0$ such that the IDS associated with the family of Fibonacci Hamiltonians satisfies

$$|N_\lambda(E_1) - N_\lambda(E_2)| \leq |E_1 - E_2|^\gamma$$

for every E_1, E_2 with $|E_1 - E_2| < \delta$.

(b) *Suppose $\lambda \geq 8$. Then for every*

$$\tilde{\gamma} > \frac{3 \log(\alpha^{-1})}{2 \log \left(\frac{1}{2} \left((\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12} \right) \right)}$$

and every $0 < \delta < 1$, there are E_1, E_2 with $0 < |E_1 - E_2| < \delta$ such that

$$|N_\lambda(E_1) - N_\lambda(E_2)| \geq |E_1 - E_2|^{\tilde{\gamma}}.$$

Thus the optimal Hölder exponent is asymptotically $\frac{3 \log(\alpha^{-1})}{2 \log \lambda}$ in the large coupling regime.

The proof is based on the self-similarity of the spectrum and an analysis of the periodic approximants (in the spirit of the proof of Theorem 5.7).

In the small coupling regime, we have the following [51]:

Theorem 5.12. *The integrated density of states $N_\lambda(\cdot)$ is Hölder continuous with Hölder exponent γ_λ , where $\gamma_\lambda \rightarrow \frac{1}{2}$ as $\lambda \rightarrow 0$, and $\gamma_\lambda < \frac{1}{2}$ for small $\lambda > 0$.*

More precisely:

(a) *For any $\gamma \in (0, \frac{1}{2})$, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, there exists $\delta > 0$ such that*

$$|N_\lambda(E_1) - N_\lambda(E_2)| \leq |E_1 - E_2|^\gamma$$

for every E_1, E_2 with $|E_1 - E_2| < \delta$;

(b) *For any sufficiently small $\lambda > 0$, there exists $\tilde{\gamma} = \tilde{\gamma}(\lambda) < \frac{1}{2}$ such that for every $\delta > 0$, there are E_1, E_2 with $0 < |E_1 - E_2| < \delta$ and*

$$|N_\lambda(E_1) - N_\lambda(E_2)| \geq |E_1 - E_2|^{\tilde{\gamma}}.$$

The proof uses the trace map formalism and a relation between the IDS of $H_{\lambda,\omega}$ and the measure of maximal entropy for the trace map T_λ . Namely, the density of states measure is proportional to the projection (along the stable manifolds) to ℓ_λ of the normalized restriction of the measure of maximal entropy $\mu_{\max}(T_\lambda)$ to an element of the Markov partition. After that, the proof uses a comparison of expansion rates of T_λ and T_0 (and is reminiscent of the proof of Hölder continuity of conjugacies between two hyperbolic dynamical systems).

Another interesting feature of the Fibonacci Hamiltonian is the uniform scaling of the density of states measure. Namely, the following holds [50]:

Theorem 5.13. *There exists $0 < \lambda_0 \leq \infty$ such that for $\lambda \in (0, \lambda_0)$, there is $d_\lambda \in (0, 1)$ so that the density of states measure dN_λ is of exact dimension d_λ , that is, for dN_λ -almost every $E \in \mathbb{R}$, we have*

$$\lim_{\varepsilon \downarrow 0} \frac{\log N_\lambda(E - \varepsilon, E + \varepsilon)}{\log \varepsilon} = d_\lambda.$$

Moreover, in $(0, \lambda_0)$, d_λ is a C^∞ function of λ , and

$$\lim_{\lambda \downarrow 0} d_\lambda = 1.$$

The proof is based on the relation between dN_λ and $\mu_{\max}(T_\lambda)$, and the exact dimensionality of hyperbolic measures [8, 108, 133].

The Hausdorff dimension of the spectrum is an upper bound for d_λ , but a priori it is not clear whether these numbers must coincide. Barry Simon conjectured that for a large class of models these quantities must be different. The next result shows that this conjecture is true for the weakly coupled Fibonacci Hamiltonian [50].

Theorem 5.14. *For $\lambda > 0$ sufficiently small, we have $d_\lambda < \dim_H \Sigma_\lambda$.*

The proof is based on the comparison of the measure of maximal entropy for T_λ (which is “responsible” for d_λ) and the equilibrium measure for the potential given by minus the log of the expansion rate. The Hausdorff dimension of the unstable projection of the latter is equal to $\dim_H \Sigma_\lambda$, and the thermodynamical description of this measure (see [122]) implies that for any other ergodic invariant measure, the dimension of its unstable projection is strictly smaller. In order to prove that those two measures are actually different, one uses the fact that the measure of maximal entropy is an equilibrium measure that corresponds to zero potential. Therefore it is enough to show that the two potentials under consideration are not cohomological,

which can be done using a comparison of multipliers of different periodic orbits of T_λ .

5.6. Gap Opening and Gap Labeling. The spectrum Σ_λ jumps from being an interval for $\lambda = 0$ to being a zero-measure Cantor set for $\lambda > 0$. Hence, as the potential is turned on, a dense set of gaps opens immediately. It is natural to ask about the size of these gaps; see [12]. These gap openings were studied in [9] for the Thue-Morse potential (where the gaps open as a power of λ) and in [11] for the period doubling potential (where some gaps open linearly, and some others are superexponentially small in λ). In the Fibonacci case, all gaps open linearly [49]:

Theorem 5.15. *For $\lambda > 0$ sufficiently small, the boundary points of a gap in the spectrum Σ_λ depend smoothly on the coupling constant λ . Moreover, given any one-parameter continuous family $\{U_\lambda\}_{\lambda>0}$ of gaps of Σ_λ , we have that*

$$\lim_{\lambda \rightarrow 0} \frac{|U_\lambda|}{|\lambda|}$$

exists and belongs to $(0, \infty)$.

Theorem 5.15 follows again from dynamical properties of the trace map. Namely, each singularity of the Cayley cubic S_0 gives birth to two periodic points on the surface S_λ , $\lambda > 0$. The distance between the periodic points is of order λ . The stable manifolds of these periodic points “cut” gaps in ℓ_λ that correspond to gaps in the spectrum. The curves formed by the families of the periodic points are normally hyperbolic manifolds of the trace map, and hence (see [86, 135]) their strong stable manifolds form a C^1 foliation. This implies that the size of each gap is also of order λ (as $\lambda \rightarrow 0$), and Theorem 5.15 follows.

The limit in Theorem 5.15 certainly depends on the family of gaps chosen. In order to study this dependence, one needs to use some labeling of the gaps. As is well known, the density of states produces such a gap labeling. That is, one can identify a canonical set of gap labels, which is only associated with the underlying dynamics (in this case, an irrational rotation of the circle or the shift-transformation on a substitution-generated subshift over two symbols), in such a way that the value of $N(E, \lambda)$ for $E \in \mathbb{R} \setminus \Sigma_\lambda$ must belong to this canonical set. In the Fibonacci case, this set is well-known (see, e.g., [12, Eq. (6.7)]) and the general gap labeling theorem specializes to the following statement:

$$(15) \quad \{N(E, \lambda) : E \in \mathbb{R} \setminus \Sigma_\lambda\} \subseteq \{\{m\alpha\} : m \in \mathbb{Z}\} \cup \{1\}$$

for every $\lambda \neq 0$. Here $\{m\alpha\}$ denotes the fractional part of $m\alpha$, that is, $\{m\alpha\} = m\alpha - \lfloor m\alpha \rfloor$. Notice that the set of gap labels is indeed λ -independent and only depends on the value of α from the underlying circle rotation. Since α is irrational, the set of gap labels is dense. In general, a dense set of gap labels is indicative of a Cantor spectrum and hence a common (and attractive) stronger version of proving Cantor spectrum is to show that the operator “has all its gaps open.” For example, the Ten Martini Problem for the almost Mathieu operator is to show Cantor spectrum, while the Dry Ten Martini Problem is to show that all labels correspond to gaps in the spectrum. The former problem has been completely solved [4], while the latter has not yet been completely settled. Indeed, it is in general a hard problem to show that all labels given by the gap labeling theorem correspond to gaps, and there are only few results of this kind. It turns out that the

stronger (or “dry”) form of Cantor spectrum holds for the weakly coupled Fibonacci Hamiltonian [49]:

Theorem 5.16. *There is $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0]$, all gaps allowed by the gap labeling theorem are open. That is,*

$$(16) \quad \{N(E, \lambda) : E \in \mathbb{R} \setminus \Sigma_\lambda\} = \{\{m\alpha\} : m \in \mathbb{Z}\} \cup \{1\}.$$

Complete gap labeling for the strongly coupled Fibonacci Hamiltonian was shown by Raymond in [136], where he proves (16) for $\lambda > 4$. We conjecture that (16) holds for every $\lambda > 0$.

Using the gap labeling, we can refine the statement of Theorem 5.15. For $m \in \mathbb{Z} \setminus \{0\}$, denote by $U_m(\lambda)$ the gap of Σ_λ where the integrated density of states takes the value $\{m\alpha\}$. Then, the following result from [49] holds:

Theorem 5.17. *There is a finite constant C^* such that for every $m \in \mathbb{Z} \setminus \{0\}$,*

$$\lim_{\lambda \rightarrow 0} \frac{|U_m(\lambda)|}{|\lambda|} = \frac{C_m}{|m|}$$

for a suitable $C_m \in (0, C^*)$.

To see why Theorem 5.17 holds, notice that each family of gaps converges (as $\lambda \rightarrow 0$) to a point of intersection of ℓ_0 with a stable manifold of one of the singularities. The intersections that have larger labels are in a sense “produced” from intersections with smaller labels by the action of the inverse of the trace map. For gaps with small labels, we know from Theorem 5.15 that $\lim_{\lambda \rightarrow 0} \frac{|U_m(\lambda)|}{|\lambda|} < C^*$ for some constant $C^* > 0$. The length (in coordinates on the two-torus covering S_0) of the piece of the stable manifold from the singularity to the point of intersection after k applications of the map is of order $\left(\frac{1+\sqrt{5}}{2}\right)^k \sim |m|$, and the contraction that will be applied to the gap is of order

$$\left(\frac{\sqrt{5}-1}{2}\right)^k \sim \left(\frac{\sqrt{5}-1}{2}\right)^{\frac{\log |m|}{\log\left(\frac{1+\sqrt{5}}{2}\right)}} = \frac{1}{|m|}.$$

5.7. Square and Cubic Fibonacci Hamiltonians. Since spectral questions for Schrödinger operators in two (and higher) dimensions are hard to study, it is natural to consider a model where known one-dimensional results can be used. In particular, let us consider the Schrödinger operator

$$\begin{aligned} [H_\lambda^{(2)}\psi](m, n) = & \psi(m+1, n) + \psi(m-1, n) + \psi(m, n+1) + \psi(m, n-1) + \\ & + \lambda (\chi_{[1-\alpha, 1)}(m\alpha \bmod 1) + \chi_{[1-\alpha, 1)}(n\alpha \bmod 1)) \psi(m, n) \end{aligned}$$

in $\ell^2(\mathbb{Z}^2)$. The theory of tensor products of Hilbert spaces and operators then implies that $\sigma(H_\lambda^{(2)}) = \Sigma_\lambda + \Sigma_\lambda$. This operator and its spectrum have been studied numerically and heuristically by Even-Dar Mandel and Lifshitz in a series of papers [73, 74, 75] (a similar model was studied by Sire in [144]). Their study suggested that at small coupling, the spectrum of $\Sigma_\lambda + \Sigma_\lambda$ is not a Cantor set; quite on the contrary, it has no gaps at all.

It turns out that this is indeed the case [49]:

Theorem 5.18. *For $\lambda > 0$ sufficiently small, $\sigma(H_\lambda^{(2)}) = \Sigma_\lambda + \Sigma_\lambda$ is an interval.*

The same statement holds for the cubic Fibonacci Hamiltonian (i.e., the analogously defined Schrödinger operator in $\ell^2(\mathbb{Z}^3)$ with spectrum $\Sigma_\lambda + \Sigma_\lambda + \Sigma_\lambda$). Theorem 5.18 follows from the estimates for the thickness of Σ_λ from Theorem 5.9 and Newhouse's Gap Lemma (Theorem 5.4). Section 7.3 shows numerical illustrations of the finite approximations $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ and $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$, along with an exploration of the number of disjoint intervals that make up these sets.

5.8. Transport Properties. There is a substantial number of papers that investigate the transport exponents associated with the Fibonacci Hamiltonian; see, for example, [15, 38, 43, 46, 52, 60, 61, 62, 63, 64, 96, 101]. While we won't describe all the known results, we want to at least highlight some of them and put them in perspective. As pointed out earlier, one of the fascinating features of quasicrystal models is that their aperiodic order being intermediate between periodic and random is reflected in a number of ways, be it in a spectral way (by spectral measures being purely singular continuous) or in a transport behavior way. Here we want to address the latter point. All the papers listed above have to goal of proving estimates that show that the transport properties of the Fibonacci Hamiltonian are markedly different from those of periodic or random media.

Since there is ballistic transport (all transport exponents are equal to one) in the periodic case and no transport (all transport exponents are equal to zero) in the random case, one therefore wants to show that the transport exponents take values in the open interval $(0, 1)$. Proving non-trivial lower bounds turns out to be comparatively easier and was accomplished in the late 1990's [38, 96] for zero phase. Several subsequent papers then went on to extend the lower bound to all phases and improved the estimates [46, 52, 60, 61, 62, 64]. Upper bounds for transport exponents, on the other hand, proved to be elusive for some time. Note a key difference here: to bound transport exponents from below, one "only" has to show that some portion of the wave packet moves sufficiently fast. On the other hand, to bound transport exponents from above, one essentially has to control the entire wave packet and show that it does not move too fast (i.e., ballistically). Thus, it is potentially easier to prove upper bounds on transport that are dual to the type of lower bound that had been established, and this indeed turned out to be the case. The papers [43, 101] showed that at least some non-trivial portion of the wave packet moves slowly. Full control and hence genuine upper bounds for transport exponents were finally obtained in 2007 and later [15, 63, 64].

Let us now state some of the transport results explicitly. Some general remarks that should be made are the following:

- (a) Almost all results concern time-averaged quantities (i.e., the exponents $\tilde{\beta}^\pm(p)$ defined in Section 4.6).
- (b) Most papers focus on the case $\psi(0) = \delta_0$. We will limit our attention here to this case as well.
- (c) The optimality of the known estimates improves when p and/or λ are large. In particular, the bounds are known to be tight in the limit $\lambda, p \uparrow \infty$.
- (d) For finite values of λ and p , the method of choice to obtain the best known bound varies.
- (e) For λ and p large enough, the transport exponent may exceed the dimension of the spectrum.

Here is a result from [62] that establishes the best known estimates for zero phase and given λ and p :

Theorem 5.19. *Suppose $\lambda > 0$ and set*

$$\gamma = D \log(2 + \sqrt{8 + \lambda^2})$$

(where D is some universal constant) and

$$\kappa = \log \left[\frac{\sqrt{17}}{20 \log(1 + \alpha)} \right].$$

Then, the time-averaged transport exponent corresponding to the initial state $\psi(0) = \delta_0$ and zero-phase Fibonacci Hamiltonian $H_{\lambda,0}$ obey

$$(17) \quad \tilde{\beta}^\pm(p) \geq \begin{cases} \frac{p+2\kappa}{(p+1)(\gamma+\kappa+1/2)}, & p \leq 2\gamma + 1; \\ \frac{1}{\gamma+1}, & p > 2\gamma + 1. \end{cases}$$

Here is a result from [63, 64] that concerns the regime of large λ and p :

Theorem 5.20. *Consider the Fibonacci Hamiltonian $H_{\lambda,\omega}$ and the initial state $\psi(0) = \delta_0$. For $\lambda > \sqrt{24}$, we have*

$$\tilde{\alpha}_u^\pm \geq \frac{2 \log(1 + \alpha)}{\log(2\lambda + 22)},$$

and for $\lambda \geq 8$, we have

$$\tilde{\alpha}_u^\pm \leq \frac{2 \log(1 + \alpha)}{\log \left(\frac{1}{2} \left[(\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12} \right] \right)}.$$

Both estimates holds uniformly in ω . In particular,

$$\lim_{\lambda \rightarrow \infty} \tilde{\alpha}_u^\pm \cdot \log \lambda = 2 \log(1 + \alpha),$$

and convergence is uniform in ω .

In fact, the upper bound can be proved also for the non-time-averaged quantities, as shown in [64].

Theorem 5.21. *Consider the Fibonacci Hamiltonian $H_{\lambda,\omega}$ and the initial state $\psi(0) = \delta_0$. For $\lambda \geq 8$ and uniformly in ω , we have*

$$\alpha_u^\pm \leq \frac{2 \log(1 + \alpha)}{\log \left(\frac{1}{2} \left[(\lambda - 4) + \sqrt{(\lambda - 4)^2 - 12} \right] \right)}.$$

6. STURMIAN POTENTIALS

The Fibonacci potential is a special case of a Sturmian potential. The latter are obtained if α in the definition of the potential, $V(n) = \lambda \chi_{[1-\alpha,1)}(n\alpha + \omega \bmod 1)$, is a general irrational number in $(0, 1)$. The Fibonacci case corresponds to the choice $\alpha = \frac{\sqrt{5}-1}{2}$.

Given an irrational $\alpha \in (0, 1)$, consider its continued fraction expansion

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

with uniquely determined $a_k \in \mathbb{Z}_+$. Truncating the continued fraction expansion of α after k steps yields the rational number p_k/q_k , which is the best rational approximant of α with denominator bounded by $q_{k+1} - 1$. The following recursions hold:

$$\begin{aligned} p_{k+1} &= a_{k+1}p_k + p_{k-1}, & p_0 &= 0, \quad p_1 = 1, \\ q_{k+1} &= a_{k+1}q_k + q_{k-1}, & q_0 &= 1, \quad q_1 = a_1. \end{aligned}$$

(In the Fibonacci case $\alpha = \frac{\sqrt{5}-1}{2}$, we have $a_k \equiv 1$ and $p_k/q_k = F_{k-1}/F_k$.) A number of the results for the Fibonacci Hamiltonian described in the previous section have been generalized to the Sturmian case under suitable assumptions on the continued fraction coefficients $\{a_k\}$. In this section, we explain what these results are, and how the proofs had to be modified.

6.1. Extension of the Trace Map Formalism. Let us denote the discrete Schrödinger operator on $\ell^2(\mathbb{Z})$ with potential $V(n) = \lambda\chi_{[1-\alpha,1)}(n\alpha + \omega \bmod 1)$ by $H_{\lambda,\alpha,\omega}$. Strong approximation again shows that the spectrum of $H_{\lambda,\alpha,\omega}$ does not depend on ω , and may therefore be denoted by $\Sigma_{\lambda,\alpha}$. The one-step transfer matrices associated with the difference equation $H_{\lambda,\alpha,\omega}u = Eu$ are given by

$$T_{\lambda,\alpha,\omega}(m, E) = \begin{pmatrix} E - \lambda\chi_{[1-\alpha,1)}(m\alpha + \omega \bmod 1) & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrices

$$M_{-1}(E) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}, \quad M_0(E) = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$M_k(E) = T_{\lambda,\alpha,0}(q_k, E) \times \cdots \times T_{\lambda,\alpha,0}(1, E) \quad \text{for } k \geq 1$$

obey the recursive relations

$$M_{k+1}(E) = M_{k-1}(E)M_k(E)^{a_{k+1}}$$

for $k \geq 0$; see [13, Proposition 1]. Passing to the variables

$$x_k(E) = \frac{1}{2}\text{Tr}M_k(E),$$

this in turn implies via the Cayley-Hamilton theorem that $x_{k+1}(E)$ can be expressed as an explicit function of (suitable Chebyshev polynomials applied to) $x_k(E), x_{k-1}(E), x_{k-2}(E)$ for $k \geq 1$; see [13, Proposition 2]. These recursion relations exhibit the same conserved quantity as before; namely, with

$$\tilde{x}_{k+1}(E) = \frac{1}{2}\text{Tr}(M_k(E)M_{k-1}(E)),$$

we have

$$\tilde{x}_{k+1}(E)^2 + x_k(E)^2 + x_{k-1}(E)^2 - 2\tilde{x}_{k+1}(E)x_k(E)x_{k-1}(E) - 1 = \frac{\lambda^2}{4}$$

for every $k \geq 0$; see [13, Proposition 3].

6.2. Results Obtained via an Analysis of the Trace Recursions. Notice that the key difference with the Fibonacci case is that, in general, the sequence of traces may not be obtained by iterating a single map. In this sense, there is in general no direct analog of the trace map. However, as we have just seen, the underlying structure of recursive relations extends nicely. The substitute for the dynamical analysis of the Fibonacci trace map will have to lie in studying the dynamics of an initial point under the successive application of a sequence of maps, the elements of which are dictated by the continued fraction expansion of α . These developments are still in their early stages. In the following we will concentrate on the known results that can be established by simply exploiting the recursive relations, without employing sophisticated tools from dynamical systems theory.

The first result that establishes a clean analogy with the Fibonacci case is the following analog of Theorem 5.1, which was established in [13].

Theorem 6.1. *Fix $\lambda > 0$ and $\alpha \in (0, 1)$ irrational. An energy E belongs to the spectrum $\Sigma_{\lambda, \alpha}$ if and only if the sequence $\{x_k(E)\}$ is bounded.*

The proof of Theorem 6.1 follows the same line of reasoning as the proof of Theorem 5.1, which was outlined in the previous section. In particular, one obtains canonical covers of the spectrum, which are useful in the estimation of its dimension. Let us make this explicit. As before, define the sets

$$\sigma_{\lambda, \alpha, k} = \{E \in \mathbb{R} : |x_k(E)| \leq 1\}$$

and

$$\Sigma_{\lambda, \alpha, k} = \sigma_k \cup \sigma_{k+1}.$$

The same reasoning shows that the sets $\Sigma_{\lambda, \alpha, k}$ are decreasing in k and the spectrum is the limiting set, that is,

$$\Sigma_{\lambda, \alpha} = \bigcap_{k \geq 1} \Sigma_{\lambda, \alpha, k};$$

see [13, Proposition 4].

A refinement of this description of the spectrum in the Sturmian case due to Raymond [136] allowed Liu and Wen to obtain the following estimates for the Hausdorff dimension of the spectrum in the large coupling regime [118].

Theorem 6.2. *Suppose $\lambda > 20$ and $\alpha \in (0, 1)$ is irrational with continued fraction coefficients $\{a_k\}$. Denote*

$$M = \liminf_{k \rightarrow \infty} (a_1 \cdots a_k)^{1/k} \in [1, \infty].$$

(a) *If $M = \infty$, then $\dim_H \Sigma_{\lambda, \alpha} = 1$.*

(b) *If $M < \infty$, then $\dim_H \Sigma_{\lambda, \alpha}$ belongs to the open interval $(0, 1)$ and obeys the estimates*

$$\dim_H \Sigma_{\lambda, \alpha} \leq \frac{2 \log M + \log 3}{2 \log M - \log \frac{3}{\lambda-8}}$$

and

$$\dim_H \Sigma_{\lambda, \alpha} \geq \max \left\{ \frac{\log 2}{10 \log 2 - 3 \log \frac{1}{4(\lambda-8)}}, \frac{\log M - \log 3}{\log M - \log \frac{1}{12(\lambda-8)}} \right\}.$$

A study of the box counting dimension of $\Sigma_{\lambda,\alpha}$ was carried out in the follow-up paper [76] by Fan, Liu, and Wen. Among other things, they showed that for $\lambda > 20$, the Hausdorff dimension and the box counting dimension of $\Sigma_{\lambda,\alpha}$ coincide whenever the sequence $\{a_k\}$ is eventually periodic.

The transport exponents in the Sturmian case were studied in the papers [38, 52, 60, 62, 121]. The following result from [62] gives dynamical lower bounds for all values of λ and p , provided α has bounded continued fraction coefficients.

Theorem 6.3. *Suppose $\lambda > 0$ and $\alpha \in (0, 1)$ is irrational with $a_k \leq C$. With*

$$\gamma = D \log(2 + \sqrt{8 + \lambda^2}) \cdot \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n a_k$$

(where D is some universal constant) and

$$\kappa = \frac{\log(\sqrt{17}/4)}{(C+1)^5},$$

the transport exponents associated with the operator $H_{\lambda,\alpha,0}$ and the initial state $\psi(0) = \delta_0$ obey

$$\tilde{\beta}^-(p) \geq \begin{cases} \frac{p+2\kappa}{(p+1)(\gamma+\kappa+1/2)}, & p \leq 2\alpha + 1; \\ \frac{1}{\gamma+1}, & p > 2\alpha + 1. \end{cases}$$

The following result from [121] gives dynamical upper bounds in the large coupling regime.

Theorem 6.4. *Suppose $\lambda > 20$ and $\alpha \in (0, 1)$ is irrational with continued fraction coefficients $\{a_k\}$ and corresponding rational approximants $\{p_k/q_k\}$. Denote*

$$D = \limsup_{k \rightarrow \infty} \frac{1}{k} \log q_k.$$

Then, the transport exponents associated with the operator $H_{\lambda,\alpha,0}$ and the initial state $\psi(0) = \delta_0$ obey

$$\tilde{\alpha}_u^\pm \leq \frac{2D}{\log \frac{\lambda-8}{3}}.$$

Moreover, if $a_k \geq 2$ for all k , then

$$\tilde{\alpha}_u^\pm \leq \frac{D}{\log \frac{\lambda-8}{3}}.$$

7. NUMERICAL RESULTS AND COMPUTATIONAL ISSUES

In this section, we provide numerical illustrations of a number of the results described in this survey. These calculations focus on the Fibonacci Hamiltonian, though many could readily be adapted to the Sturmian potentials described in the last section. We begin by studying approximations to the spectrum for the Fibonacci model in one dimension, then investigate estimates of the integrated density of states based on spectra of finite sections of the operator. Finally, we address upper bounds on the spectrum in two and three dimensions. In all cases, we set the phase ω to zero.

7.1. Spectral Approximations for the Fibonacci Hamiltonian. We begin by calculating the spectrum σ_k for the k th periodic approximations to the Fibonacci potential. The analysis described in Section 5 suggests several ways to compute σ_k , which turn out to have varying degrees of utility.

Given a candidate energy E , one can test if $E \in \sigma_k$ by iterating the trace recurrence (6) and testing if $|x_k(E)| \leq 1$. In principle, this simple approach enables investigation for arbitrarily large values of k . However, two key obstacles restrict the utility of this method: (i) it does not readily yield the entire set σ_k ; (ii) as k increases, the intervals that comprise σ_k become exponentially narrow, beyond the resolution of the standard floating point number system in which such calculations are typically performed. However, this approach can yield some useful results, particularly in the small coupling regime where the decay of the interval widths is most gradual, or when one is only interested in some narrow set of energy values. (This method of calculation was used to produce illustrations in [49].)

To obtain the entire set σ_k , one might instead use the recurrence (6) to construct the degree- F_k polynomial $x_k(E)$, then determine the regions where $|x_k(E)| \leq 1$ by finding the zeros of the polynomials $x_k(E) + 1$ and $x_k(E) - 1$ using a standard root-finding algorithm. For all but the smallest k this approach is untenable. Coefficients of $x_k(E)$ grow exponentially in k ; e.g., for $\lambda = 4$,

$$\begin{aligned} x_6(E) = & \frac{1}{2}E^{13} - 16E^{12} + \frac{435}{2}E^{11} - 1616E^{10} + \frac{13905}{2}E^9 - 16272E^8 + 13330E^7 \\ & + 20160E^6 - 37133E^5 - 17056E^4 + \frac{61013}{2}E^3 + 25104E^2 + \frac{13021}{2}E + 560. \end{aligned}$$

The magnitude of these coefficients, compounded by the proximity of the roots for larger values of λ and k , leads to inaccurate root calculations, a phenomenon well studied by numerical analysts; see, e.g. [124, 151]. Indeed, it is not uncommon for the computed roots to be so inaccurate as to have significant spurious imaginary parts.

There is a more robust approach to computing the approximate Fibonacci spectrum σ_k . One can view σ_k as the exact spectrum of a related Schrödinger operator with a potential having period F_k . The spectrum of this operator is the union of F_k non-degenerate intervals whose end points are given by the eigenvalues of the two F_k -dimensional matrices J_{k+} and J_{k-} :

$$J_{k\pm} = \begin{pmatrix} v_{1,k} & 1 & & & \pm 1 \\ 1 & v_{2,k} & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & v_{F_k-1,k} & 1 \\ \pm 1 & & & 1 & v_{F_k,k} \end{pmatrix},$$

with unspecified entries beyond the tridiagonal section equal to zero; see, e.g., [149, Ch. 7]. Here the potential values $v_{n,k}$ are given by

$$(18) \quad v_{n,k} = \lambda \chi_{[1-F_{k-1}/F_k, 1)}(nF_{k-1}/F_k \bmod 1).$$

This approach, which we use for the computations described below, has also been employed in the context of Fibonacci computations by Even-Dar Mandel and Lifshitz [73].

The standard procedure for computing all the eigenvalues of a symmetric matrix begins by applying a unitary similarity transformation to reduce the matrix to

symmetric tridiagonal form.⁸ Floating-point arithmetic introduces errors into this process, resulting in the exact tridiagonal reduction of a matrix that differs from the intended matrix by a factor that scales with the precision of the floating point arithmetic system, the coupling constant λ , and the dimension F_k . The eigenvalues of this tridiagonal matrix are then approximated to high accuracy via a procedure known as QR iteration [131]. Remarkably, this iteration does not introduce significant errors beyond those incurred by the reduction to tridiagonal form; for a discussion of this accuracy, see [1, 152]. Overall, this process requires $O(F_k^3)$ floating point arithmetic operations and the storage of $O(F_k^2)$ floating point numbers. (The conventional procedure for reducing the matrix to tridiagonal form destroys the zero structure present in $J_{k\pm}$.) Of course, the upper estimate $\Sigma_{k,\lambda} = \sigma_{k,\lambda} \cup \sigma_{k+1,\lambda}$ then requires computation of all eigenvalues of four matrices.

Significant insight can be gleaned from numerical calculations involving small to moderate values of k . For example, Figure 3 shows $\sigma_{k,\lambda}$ for $\lambda \in [0, 2]$ and $k = 1, \dots, 8$, while Figure 4 shows the upper bounds $\Sigma_{k,\lambda}$ for the same range of λ and k . Since $\Sigma_{8,\lambda} = \sigma_{8,\lambda} \cup \sigma_{9,\lambda}$, for $\lambda > 0$ the spectrum is the union of 34 and 55 intervals.

To develop conjectures (e.g., regarding $\dim \Sigma_\lambda$), one would like use approximations to Σ_λ for larger k . Two fundamental challenges arise: (i) the $O(F_k^3)$ work and $O(F_k^2)$ storage becomes prohibitive; (ii) while non-degenerate, the intervals in $\sigma_{k,\lambda}$ become exponentially small and exponentially close together. This phenomenon is illustrated in Figure 5. The utility of the numerical results degrade when the size of these bands and gaps approaches the order of the error in the numerical computation.⁹ On contemporary commodity computers, computations of $\Sigma_{k,\lambda}$ up to roughly $k = 20$ (requiring all eigenvalues of matrices of dimension $F_{20} = 10,946$ and $F_{21} = 17,711$) is feasible, provided λ is sufficiently small for the results to be accurate. Recently Puelz has proposed an improved approach that ameliorates challenge (i) above by reducing the required work to $O(F_k^2)$ and storage to $O(F_k)$, and challenge (ii) by using extended precision arithmetic [134].

To estimate the box-counting dimension of Σ_λ (assuming it exists), we use the definition

$$\dim_B(S) = \lim_{\varepsilon \rightarrow 0} \frac{\log C_S(\varepsilon)}{\log 1/\varepsilon},$$

where $C_S(\varepsilon)$ counts the number of intervals of width ε that intersect S ,

$$C_S(\varepsilon) := \#\{j \in \mathbb{Z} : [j\varepsilon, (j+1)\varepsilon) \cap S \neq \emptyset\}.$$

Note that $\dim_B(\Sigma_{k,\lambda}) = 1$ for all k , since $\Sigma_{k,\lambda}$ is the union of finitely many closed intervals. Still, one gains insight into $\dim_B(\Sigma_\lambda)$ from $\log(C_{\Sigma_{k,\lambda}}(\varepsilon))/\log(1/\varepsilon)$ for finite values of ε and various k , as can be seen in Figure 6. For fixed λ , the resulting estimates of $\dim_B(\Sigma_\lambda)$ (taken, e.g., as $\inf_{\varepsilon \in (0,1)} \log(C_{\Sigma_{k,\lambda}}(\varepsilon))/\log(1/\varepsilon)$) apparently improve as k increases; lower values of k are suitable for larger values of λ . However, with this approach it is difficult to accurately estimate the critical value

⁸Methods such as the Lanczos algorithm excel at computing *a few* eigenvalues of large symmetric matrices [131, Ch. 13]. These methods are not feasible here, for *all* eigenvalues of $J_{k\pm}$ are required. However, if one is only interested in a narrow band of energies, these methods can be highly effective.

⁹ More subtly, the formula (18) incurs significant rounding errors for large n and k , resulting in errors on the diagonal of $J_{k\pm}$ of size λ . For greater accuracy, one should use the equivalent formulation $v_{n,k} = \lambda \chi_{[F_k - F_{k-1}, F_k)}(nF_{k-1} \bmod F_k)$, which is more robust.

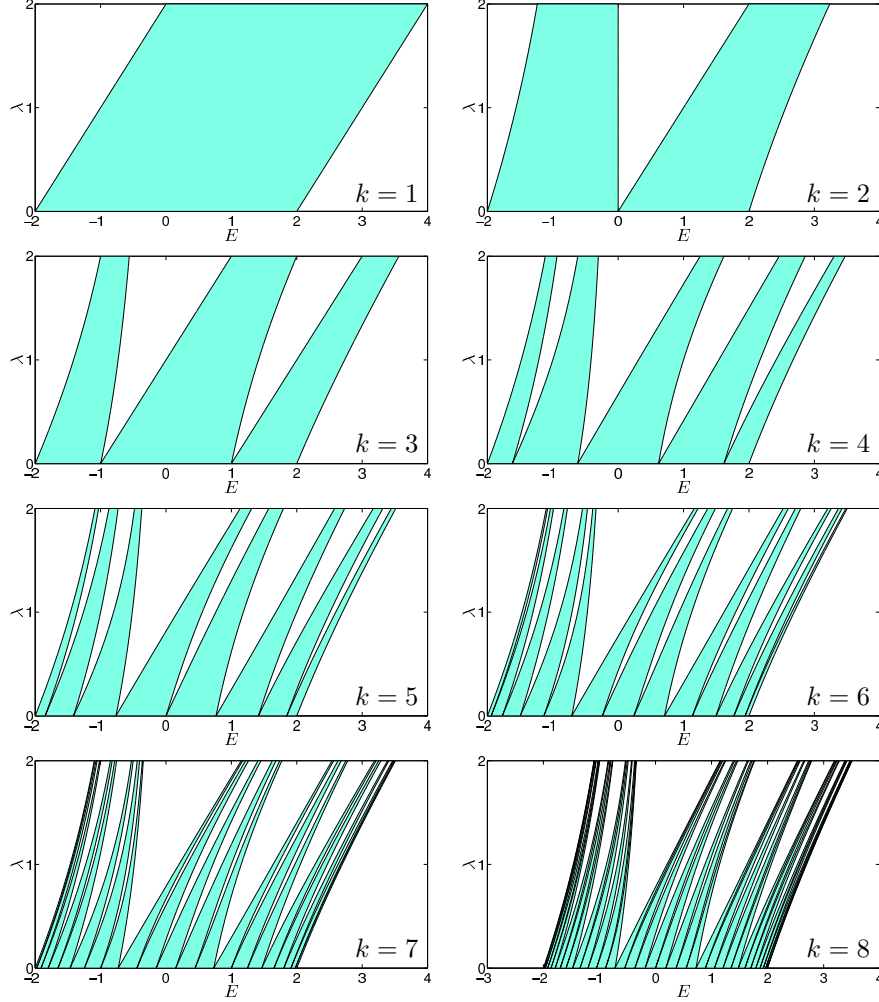


FIGURE 3. Spectra of the periodic approximations $\sigma_{k,\lambda}$ for the Fibonacci Hamiltonian, as a function of $\lambda \in [0, 2]$. For $k = 8$ and all $\lambda > 0$, $\sigma_{k,\lambda}$ is the union of $F_8 = 34$ disjoint intervals.

at which $\dim_B(\Sigma_\lambda) = 1/2$. (A rough estimate, suggested from Figure 6, is $\lambda \approx 4$; see the discussion preceding Problem 8.7 below.) More accurate approximations will require computations with larger values of k than are feasible with the method described above.

Finally, Figure 7 explores numerical computations of the thickness, defined in (10). As established in Theorem 5.9, the thickness $\tau(\Sigma_\lambda)$ behaves like $1/\lambda$ as $\lambda \downarrow 0$. As λ decreases we see this behavior mirrored in the upper bounds $\Sigma_{k,\lambda}$, up to some point where $\tau(\Sigma_{k,\lambda})$ rapidly increases: $\Sigma_{k,\lambda}$ is the union of no more than $F_k + F_{k+1}$ intervals separated by gaps that diminish as $\lambda \downarrow 0$.

7.2. Density of States for the Fibonacci Model. We next turn to an investigation of the exponent of Hölder continuity of the integrated density of states

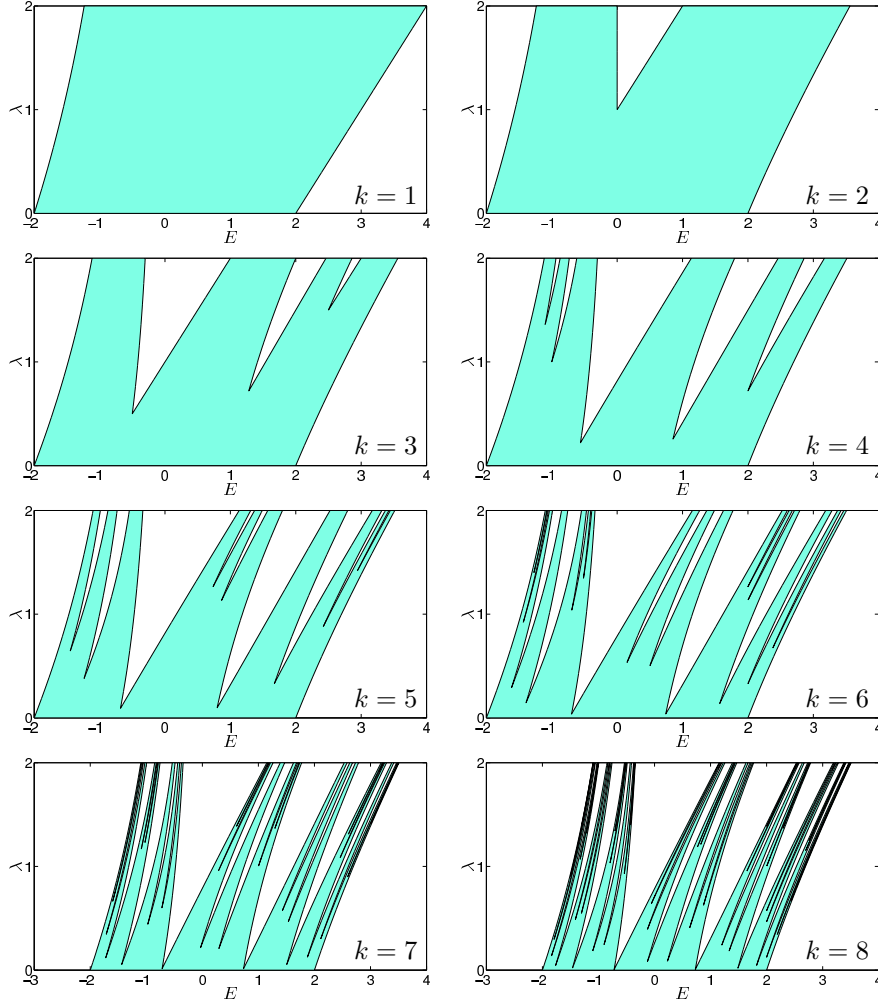


FIGURE 4. The upper bound $\Sigma_{k,\lambda} = \sigma_{k,\lambda} \cup \sigma_{k+1,\lambda}$ on the Fibonacci spectrum, as a function of $\lambda \in [0, 2]$. For $k = 8$ and $\lambda = 2$, $\Sigma_{k,\lambda}$ is the union of 42 disjoint intervals.

(IDS) for the Fibonacci model, discussed in Section 5.5. To estimate $N_\lambda(E)$ in equation (14), one must compute all the eigenvalues of $H_{\lambda,[1,n]}$, the restriction of H_λ to sites $[1, n]$ with Dirichlet boundary conditions. This restriction is an $n \times n$ tridiagonal matrix; because this matrix lacks the corner entries present in $J_{k\pm}$ in the last subsection, its eigenvalues can readily be efficiently computed for large values of n (say $n \leq 10^6$ on contemporary desktop computers). While computational complexity is no longer such a constraint, accuracy still is: for large n and λ , some eigenvalues of $H_{\lambda,[1,n]}$ are closer than the precision of the floating point arithmetic,

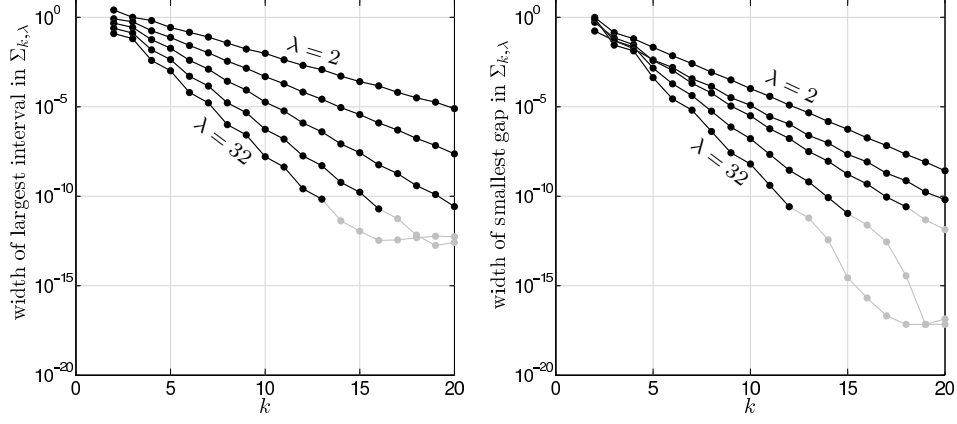


FIGURE 5. Exponential decay of the largest intervals and smallest gaps in the approximations $\Sigma_{k,\lambda}$, for coupling constants $\lambda = 2, 4, 8, 16, 32$, as computed in MATLAB's double-precision floating point arithmetic. The data points that are plotted in gray are likely dominated by computational errors.

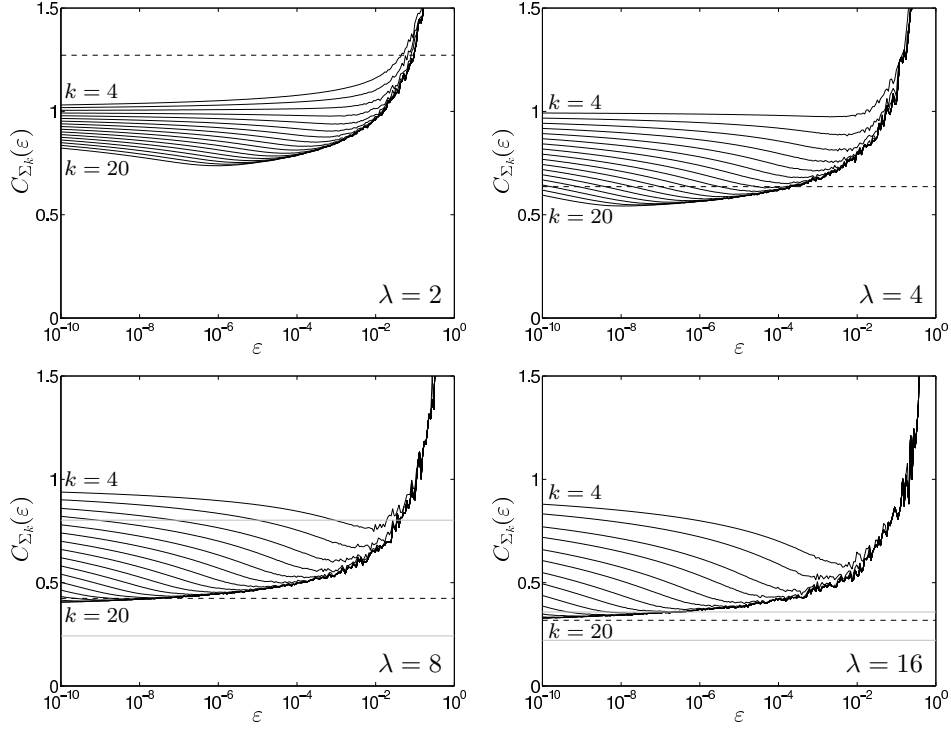


FIGURE 6. Estimates of $\dim_B(\Sigma_\lambda)$ for various values of λ , based on the upper bounds $\Sigma_{k,\lambda}$ for various k . The dashed horizontal line denotes $\log(1 + \sqrt{2})/\log(\lambda)$, to which $\dim_B(\Sigma_\lambda)$ tends as $\lambda \rightarrow \infty$ (Theorem 5.7). The gray horizontal lines in the bottom plots show the upper and lower bounds (12)–(13).

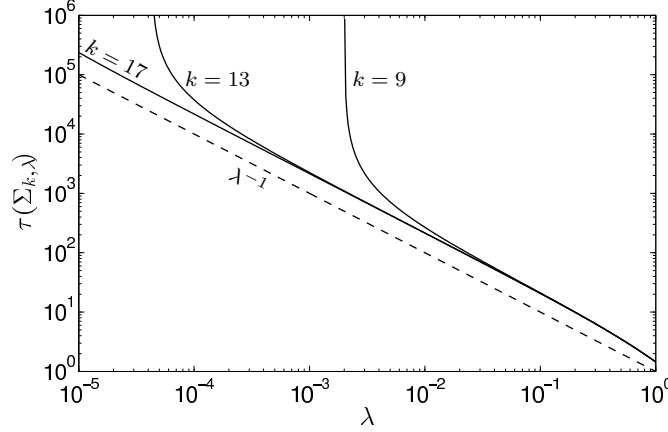


FIGURE 7. Thickness of $\Sigma_{k,\lambda}$ as a function of λ for three values of k , consistent with Theorem 5.9.

rendering, for example, $|E_1 - E_2| = 0$ for theoretically distinct eigenvalues E_1 and E_2 of $H_{\lambda,[1,n]}$.¹⁰

Figure 8 shows estimates of the IDS based on computations with $n = 10,000$ for λ values ranging from the trivial case of no coupling ($\lambda = 0$) to strong coupling ($\lambda = 8$). The fine structure of the spectrum is evident in Figure 9, which repeatedly zooms in upon subsets of the spectrum of the finite section $H_{\lambda,[1,n]}$ for $\lambda = 1$ and $n = 100,000$. (The numerical concerns discussed in the last paragraph do not affect these figures.)

We now explore the Hölder continuity of the integrated density of states. In consideration of (14), define

$$N_{n,\lambda}(E) = \lim_{n \rightarrow \infty} \frac{\#\{\text{eigenvalues of } H_{\lambda,[1,n]} \text{ that are } \leq E\}}{n}.$$

Figure 10 investigates the large λ behavior of the Hölder exponent addressed in Theorem 5.11, based on computations with finite sections of dimension $n = 10,000$. Indeed, we see asymptotic behavior like $\frac{3 \log(\alpha^{-1})}{2 \log \lambda}$, and moreover the figure suggests that the dimension of the measure is smooth in this regime.

7.3. Spectral Estimates for Square and Cubic Fibonacci Hamiltonians.

As described in Section 5.7, the estimates $\Sigma_{k,\lambda}$ for the one-dimensional Fibonacci spectrum can readily be translated into approximations for the square and cubic cases, as investigated by Even-Dar Mandel and Lifshitz [73]. As described in Theorem 5.18, Σ_λ need not be a Cantor set, especially for small coupling constants. This behavior is apparent in Figures 11 and 12, which illustrate $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ and $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ for various values of k and λ . For a finite range of small λ values, the spectra are intervals that branch into a greater number of intervals as k and λ increase. Figure 13 shows the growth in the number of intervals present in these approximations as a function of λ for three different values of k . This plot makes evident rapid (but not always monotone) growth in the number of intervals with

¹⁰By its structure, $H_{\lambda,[1,n]}$ must have n distinct eigenvalues. However, scenarios where eigenvalues are exceptionally close are well-known in the numerical analysis community; see, e.g., Wilkinson's W_{21}^+ matrix [131, Sec. 7.7].

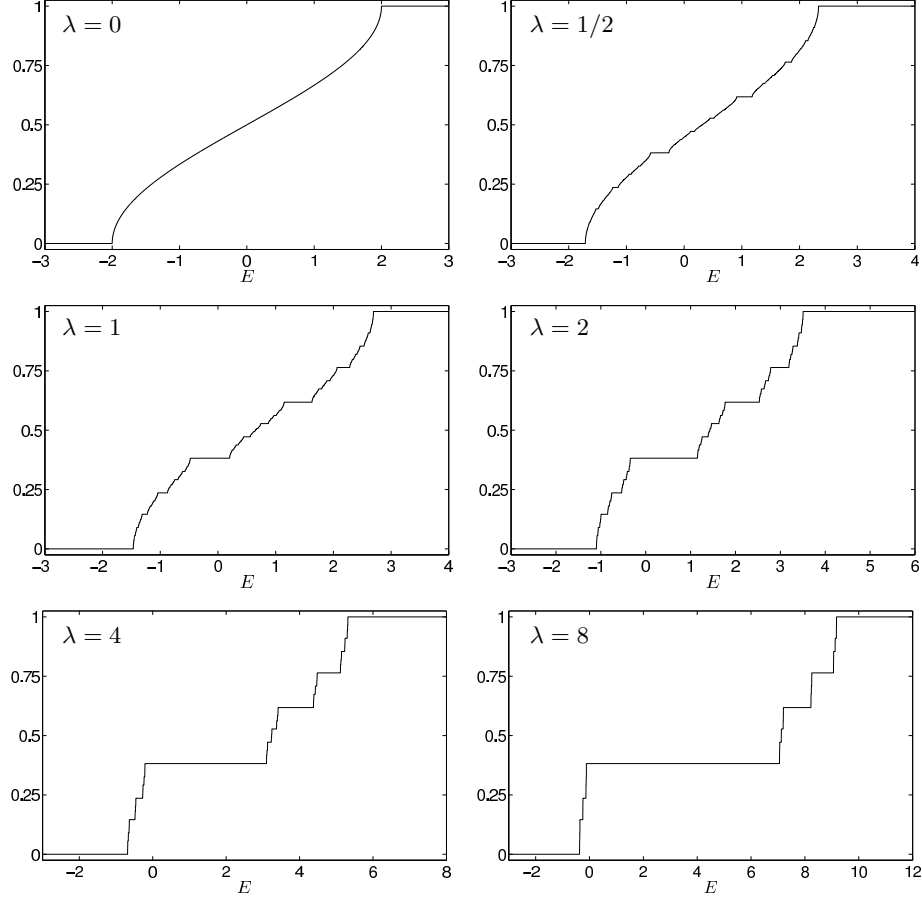


FIGURE 8. Approximations to the integrated density of states for the Fibonacci model with various values of the coupling constant, λ , based on $n = 10,000$.

λ . Figure 14 illustrates the opening and closing of gaps for the square problem, revealing an intriguing structure for finite k . How does this structure develop as k increases, and, indeed, is it reflected in $\Sigma_\lambda + \Sigma_\lambda$? At present these questions remain open.

Tables 1 and 2 investigate the square and cubic spectral estimates more precisely, giving the values of λ where multiple intervals first emerge. These results confirm and sharpen the observation of Even-Dar Mandel and Lifshitz [73] that $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ transitions from one to two intervals near $\lambda = 1.3$, while $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ makes the same transition near $\lambda = 2$. For these finite values of k , it is apparent that $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ and $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ both transition to two intervals, then three intervals, and so on. What do these calculations suggest about the limit $k \rightarrow \infty$? For example, is the λ value at which $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ transitions from two to three intervals converging? Is there a finite span of λ values for which $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ persists as the union of two intervals as $k \rightarrow \infty$, or does $\Sigma_\lambda + \Sigma_\lambda$ transition from one interval directly to a Cantorval or Cantor set? (See Problems 8.7 and 8.8 below.)

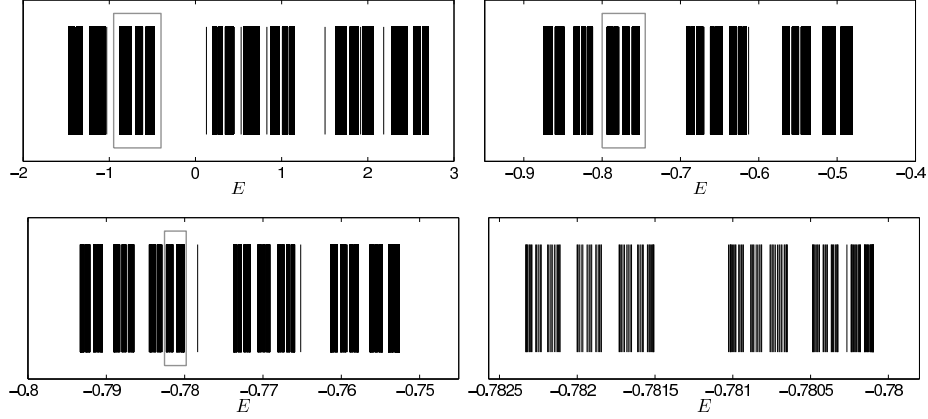


FIGURE 9. Eigenvalues of $H_{\lambda,[1,n]}$ for $\lambda = 1$ and $n = 100,000$, drawn as vertical lines to aid visibility. The first plot shows the entire spectrum; the gray boxes denote the region on which the next plot zooms.

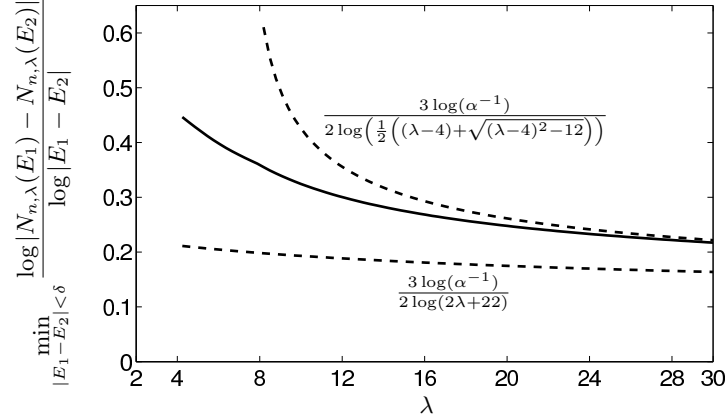


FIGURE 10. Illustration of Theorem 5.11, based on numerically-computed eigenvalues from finite sections $H_{\lambda,[1,n]}$ for $n = 10,000$. Here $\delta = 0.025$ and the minimization is over $E_1, E_2 \in \sigma(H_{\lambda,[1,n]})$.

Let $\lambda_{k,m}$ denote the value of λ at which $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ (or $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$) first splits from m to $m+1$ intervals as λ increases, with $\lambda_{k,0} = 0$. (Our detailed computations suggest that, for small values of m , there is only one such point of transition; for larger numbers of intervals, gap closings complicate the picture, as seen in Figure 14.) Figure 15 plots $\lambda_{k,m} - \lambda_{k,m-1}$ as a function of k for $m = 1, \dots, 7$ for the square and cubic Hamiltonians. Do the transition points converge as $k \rightarrow \infty$? First consider the plot on the left, for the square Hamiltonian. For $m = 1$ and $m = 2$, $\lambda_{k,m}$ appears to converge; however, the points of transition to $m \geq 3$ intervals do not show such consistency: it is unclear if these $\lambda_{k,m}$ values are converging. It may be that the coupling constants at which $\Sigma_{\lambda,k} + \Sigma_{\lambda,k}$ breaks into $m > 3$ intervals are converging to the point at which the spectrum breaks into $m = 3$ intervals as $k \rightarrow \infty$. Now consider the plot on the right of Figure 15, for the

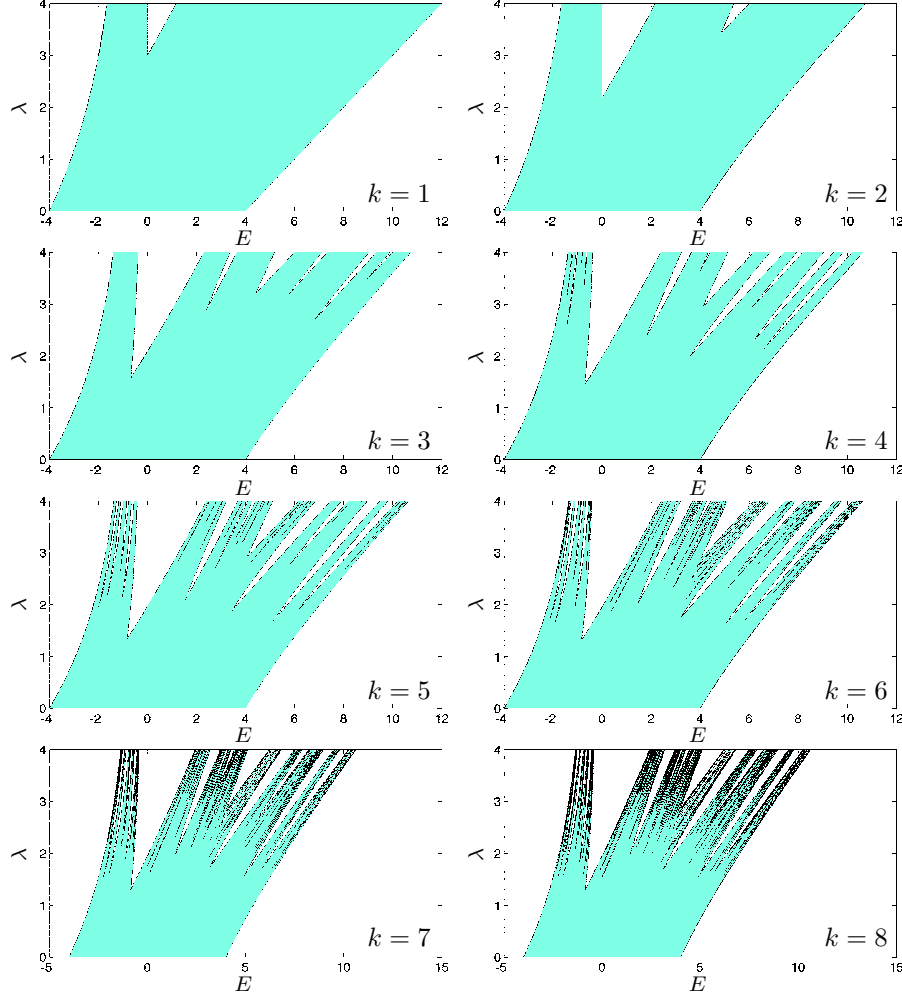


FIGURE 11. Approximations $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ to the spectrum of the two-dimensional Fibonacci operator, as a function of λ . For $k = 8$ and $\lambda = 4$, $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ is the union of 311 disjoint intervals.

cubic Hamiltonian. In contrast to the square case, these results suggest the $\lambda_{k,m}$ values converge to distinct points as $k \rightarrow \infty$ for all values $m = 1, 2, \dots, 7$ shown, inviting the conjecture that there exist λ values for which $\Sigma_\lambda + \Sigma_\lambda + \Sigma_\lambda$ is the union of m disjoint intervals for all $m \geq 1$.

8. CONJECTURES AND OPEN PROBLEMS

In this final section we discuss various open problems that are suggested by the existing results and address generalizations, strengthenings, and related issues.

We begin with open problems for the Fibonacci Hamiltonian. The existing quantitative results concern estimates for dimensional properties of the spectrum, the density of states measure, and the spectral measures, as well as estimates for the

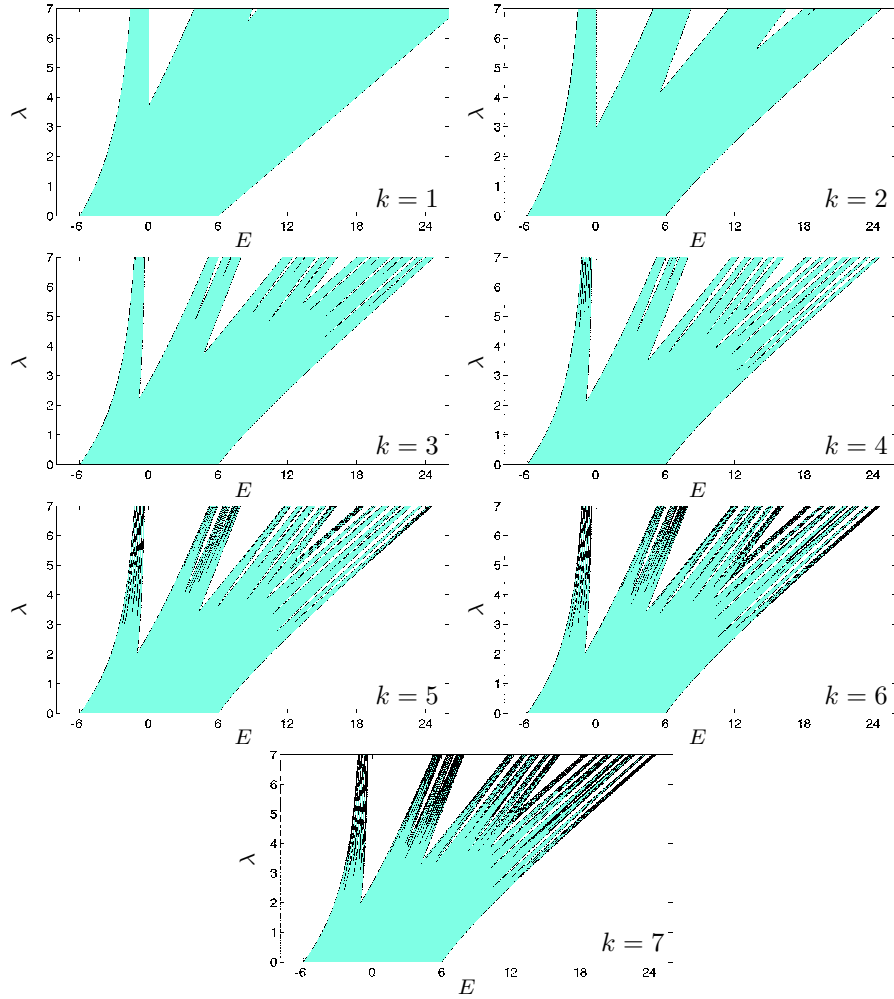


FIGURE 12. Approximations $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ to the spectrum of the three-dimensional Fibonacci operator, as a function of λ . For $k = 7$ and $\lambda = 7$, $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ is the union of 482 disjoint intervals.

transport exponents. In almost all cases, the asymptotic behavior is known in the regimes of small and large coupling. While the bounds we obtain are monotone, we would like to understand whether the quantities themselves have this property:

Problem 8.1. *Are the various quantities we consider (in particular, $\dim_H \Sigma_\lambda$) monotone in λ ?*

The known estimates for dimensional properties of the spectral measures are the only ones that are clearly not optimal, and in particular do not identify the asymptotics in the extremal coupling regimes. For the density of states measure, which is an average of spectral measures, we have much better information. Can one find ways to find equally good estimates for spectral measures?

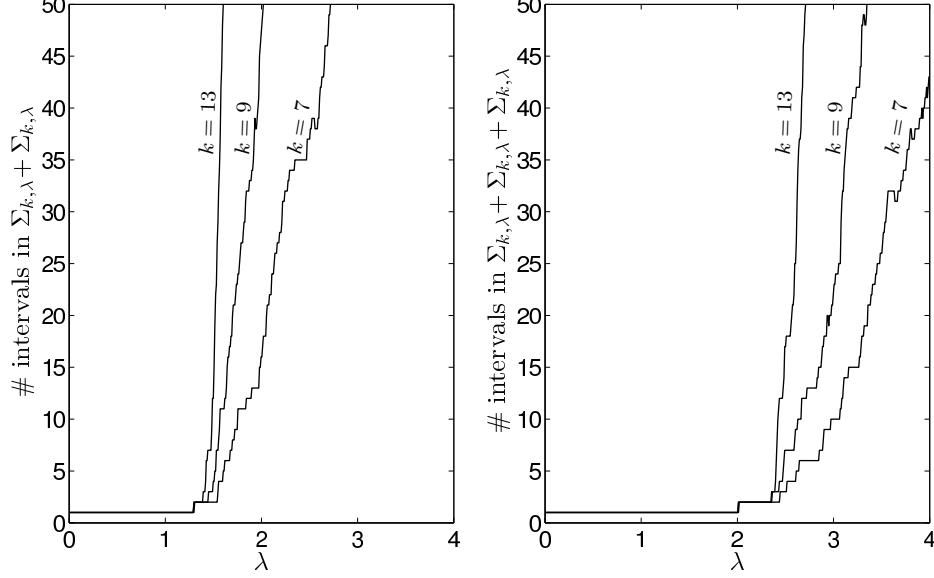


FIGURE 13. Number of intervals in the spectral approximations $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ and $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$, as a function of λ .

TABLE 1. Estimates of λ^* , the λ value for which the thickness of $\Sigma_{k,\lambda}$ equals one, along with $\lambda_{k,m}$, the coupling constant where $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ splits from m to $m + 1$ intervals, for $m = 1, \dots, 4$.

k	λ_k^*	$\lambda_{k,1}$	$\lambda_{k,2}$	$\lambda_{k,3}$	$\lambda_{k,4}$
6	1.313172936	1.313172936	1.624865906	1.649775155	1.708521471
7	1.298964798	1.298964798	1.543759898	1.548912772	1.596682038
8	1.296218739	1.296218739	1.494856217	1.514291562	1.520122025
9	1.294303086	1.294303086	1.445808095	1.492410878	1.512965310
10	1.293935333	1.293935333	1.442778219	1.446787662	1.472813609
11	1.293679331	1.293679331	1.430901095	1.436192692	1.437915282
12	1.293630242	1.293630242	1.402035016	1.415460742	1.426586813
13	1.290031553	1.293596081	1.392730451	1.412863780	1.419815054
14	1.288819456	1.293589532	1.382510414	1.404399139	1.408704405
15	1.287431935	1.293584975	1.380466052	1.399646887	1.400190389
16	1.287269802	1.293584102	1.380121550	1.388518687	1.397593470
17	1.287084388	1.293583494	1.379851608	1.387310733	1.395556145
18	1.287062735	1.293583377	1.379806139	1.385835331	1.393702258
19	1.287037977				
20	1.287035086				

Problem 8.2. *What can one say about the spectral measures? In particular, are their dimensional properties uniform across the hull and/or across the spectrum? Moreover, what are the asymptotics as $\lambda \downarrow 0$ and $\lambda \uparrow \infty$?*

We know that $\dim \Sigma_\lambda$ goes to one as λ goes to zero. In addition, we would be interested in the following:

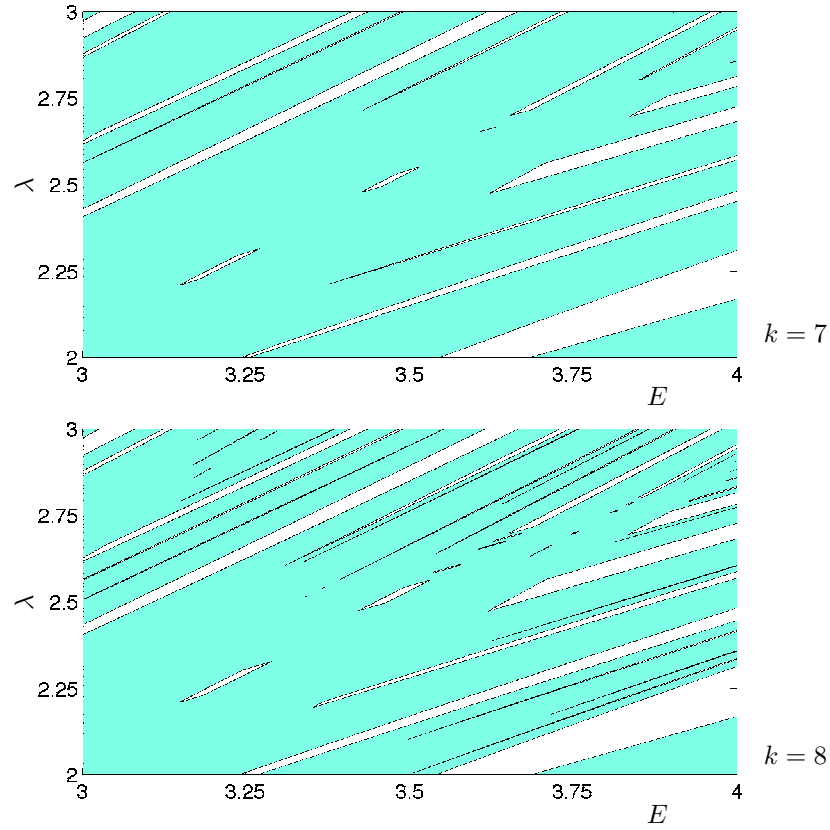


FIGURE 14. Approximations $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ to the two-dimensional Fibonacci spectrum $\Sigma_\lambda + \Sigma_\lambda$ as in Figure 11, magnified to show the opening and closing of gaps as λ increases. How this structure affects $\Sigma_\lambda + \Sigma_\lambda$ is not presently understood.

TABLE 2. Estimates of $\lambda_{k,m}$, the coupling constant where $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ splits from m to $m + 1$ intervals, for $m = 1, \dots, 4$.

k	$\lambda_{k,1}$	$\lambda_{k,2}$	$\lambda_{k,3}$	$\lambda_{k,4}$
6	2.025741216	2.544063632	2.573539294	2.842670115
7	2.012664501	2.438240772	2.511570744	2.606841186
8	2.011113604	2.376933028	2.498126298	2.498926850
9	2.009524869	2.364541039	2.435665993	2.473875055
10	2.009337409	2.357357667	2.412613336	2.421115367
11	2.009145619	2.355932060	2.399696274	2.408616763
12	2.009123008	2.355107791	2.392573154	2.401253561
13	2.009099880	2.354944739	2.391094663	2.397036745
14	2.009097154	2.354850520	2.390282080	2.393347062
15	2.009094365	2.354831891	2.390113912	2.393329303
16	2.009094036	2.354821128	2.390021550	2.393302392
17	2.009093700	2.354819000	2.390002443	2.393300376

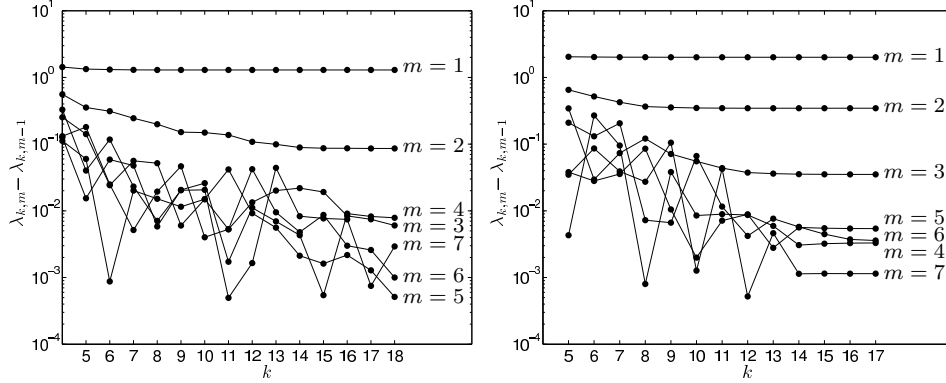


FIGURE 15. The span of λ values (i.e., $\lambda_{k,m} - \lambda_{k,m-1}$) for which $\Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ (left) and $\Sigma_{k,\lambda} + \Sigma_{k,\lambda} + \Sigma_{k,\lambda}$ (right) comprise m intervals (i.e., $\lambda_{k,m} - \lambda_{k,m-1}$) for $m = 1, \dots, 7$.

Problem 8.3. *Does the right-derivative of $\dim \Sigma_\lambda$ exist at zero?*

On a technical level, an answer to the following question is crucial:

Problem 8.4. *Show that ℓ_λ is transversal to $W^s(\Omega_\lambda)$ for all λ values.*

A proof of this property would immediately result in an extension of several results that are known to hold for small (and/or large) values of λ to all values of λ .

Let us now turn to the higher-dimensional separable analogs of the Fibonacci Hamiltonian (e.g., the square or cubic Fibonacci Hamiltonian). Recall that the spectrum of such an operator is given by the sum of the one-dimensional spectra, which in turn are Cantor sets. Recall also that at sufficiently small coupling, these sum sets are intervals, while at sufficiently large coupling, they are Cantor sets as well. Concretely, this uses that if the thickness of a Cantor set C is larger than 1, then $C + C$ is an interval by Theorem 5.4 and, on the other hand, if the upper box counting dimension of C is strictly less than $1/2$, then $C + C$ is a Cantor set. It is natural to ask what shape the higher-dimensional spectra have at intermediate coupling, that is, we wish to study how the transition from $C + C$ being an interval to being a Cantor set happens when the thickness of C decreases.

Definition 8.5. *A compact set $C \subset \mathbb{R}^1$ is a Cantorval if it has a dense interior (i.e., $\text{int}(C) = C$), it has a continuum of connected components, and none of them is isolated.*

Here is a general result on the occurrence of Cantorvals in the context of taking sums of Cantor sets [123]:

Theorem 8.6. *There is an open set \mathcal{U} in the space of dynamically defined Cantor sets such that for generic $C_1, C_2 \in \mathcal{U}$, the sum $C_1 + C_2$ is a Cantorval.*

Unfortunately, this result does not provide any specific and verifiable genericity conditions that would allow one to check that the sum of two given specific Cantor sets is indeed a Cantorval. Thus, for our purpose we need a solution to the following problem.

Problem 8.7. *Provide specific verifiable conditions on a Cantor set C which imply that the sum $C + C$ is a Cantorval.*

Ideally, such a criterion would be applicable to the spectrum of the Fibonacci Hamiltonian and establish that, say, the spectrum of the square Fibonacci Hamiltonian is a Cantorval for intermediate values of the coupling constant λ . The next step would then be to study the transitions between the three regimes. We ask whether there are two sharp transitions; compare [74] for closely related numerical evidence and discussion.

Problem 8.8. *Let $H_\lambda^{(2)}$ be the separable square Fibonacci Hamiltonian. Prove that there are values $0 < \lambda' < \lambda'' < \infty$ such that for $\lambda \in (0, \lambda')$, the spectrum $\sigma(H_\lambda)$ is an interval (or a finite union of intervals), for $\lambda \in (\lambda', \lambda'')$, it is a Cantorval, and for $\lambda \in (\lambda'', \infty)$, it is a Cantor set.*

Notice that this will provide an example of a (*topologically!*) new structure of the spectrum for “natural” potentials.

Moving on from the Fibonacci case, which has a description via a substitution rule as well as via a simple quasi-periodic expression, there are two natural choices of a more general setting.

For a different choice of the underlying substitution rule, one always has an associated trace map. However, our understanding of the dynamics of such a trace map is in general far more limited than the one in the Fibonacci case. As a consequence, outside of the Fibonacci case there is a scarcity of quantitative results for dimensional issues (such as the dimension of the spectrum, the dimension of the density of states measure, or the dimension of the spectral measures). For example, here is a simple open problem that is currently completely out of reach:

Problem 8.9. *Study other trace maps (e.g., period doubling and Thue-Morse); in particular, find the asymptotics of the Hausdorff dimension of the spectrum as the coupling constant tends to zero or infinity.*

The other natural generalization of the Fibonacci potential is to replace the golden ratio in its quasi-periodic description by a general irrational number. Thus, we discuss some open problems for Sturmian potentials next.

Let us say that two Cantor sets C_1 and C_2 on \mathbb{R}^1 are *diffeomorphic* if there are neighborhoods $U_1(C_1)$, $U_2(C_2)$, and a diffeomorphism $f : U_1 \rightarrow U_2$ such that $f(C_1) = C_2$.

Problem 8.10. *Suppose that $\alpha = [a_1, a_2, \dots]$ and $\beta = [b_1, b_2, \dots]$ are such that for some $k \in \mathbb{Z}$ and all large enough $i \in \mathbb{Z}_+$ we have $b_{i+k} = a_i$. Prove that in this case, the Sturmian spectra $\Sigma_{\lambda, \alpha}$ and $\Sigma_{\lambda, \beta}$ are diffeomorphic.*

Notice also that due to the ergodicity of the Gauss map, a solution of this problem would also imply that the following long standing conjecture is correct:

Problem 8.11. *For any fixed $\lambda > 0$, the dimension $\dim_H \Sigma_{\lambda, \alpha}$ is almost everywhere constant in α .*

Finally, let us emphasize that most of the questions related to higher dimensional models (described in Section 3) are completely open. So we formulate an extremely general problem:

Problem 8.12. *Study spectral properties (e.g., the shape of the spectrum and the type of the spectral measures) and transport properties of higher dimensional operators; for example, study these questions for the particular case of the Laplacian on the graph associated with a Penrose tiling.*

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